Linear Algebra and its Applications 439 (2013) 3490–3504

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Direct and inverse spectral problems for a class of non-self-adjoint periodic tridiagonal matrices

Natália Bebiano [a], Mikhail Tyaglov [b,*,1]

[a] *Departamento de Matematica, Universidade de Coimbra, P-3001-454 Coimbra, Portugal*
[b] *Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, PR China*

ARTICLE INFO

Article history:
Received 19 April 2013
Accepted 18 September 2013
Available online 17 October 2013
Submitted by Y. Wei

MSC:
65F15
15A18

Keywords:
Inverse eigenvalue problem
Stable polynomials
Hermite–Biehler theorem
Jacobi matrix
Periodic Jacobi matrix

ABSTRACT

The spectral properties of a class of tridiagonal matrices are investigated. The reconstruction of matrices of this special class from given spectral data is also studied. Necessary and sufficient conditions for that reconstruction are found. The obtained results extend some results on the direct and inverse spectral problems for periodic Jacobi matrices and for some non-self-adjoint tridiagonal matrices.

1. Introduction

Inverse eigenvalue problems arise in mathematics as well as in many areas of engineering and science such as chemistry, geology, physics, etc. Often the mathematical model describing a certain physical system involves matrices whose spectral data allow the prediction of the behavior of the system. Determining the spectra of those matrices is the so-called *direct problem*, while the *inverse problem* consists in the reconstruction of the matrices from the knowledge of the behavior of the system, frequently expressed by spectral data.

* Corresponding author.
E-mail addresses: bebiano@mat.uc.pt (N. Bebiano), tyaglov@sjtu.edu.cn (M. Tyaglov).
1 Supported by the International Young Scientists Fellowship of Natural Science Foundation of China.

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.09.026

Inverse eigenvalue problems, in general, and for structured matrices, in particular, have attracted attention of many researchers, some of them motivated by the numerous applications of this scientific area (see e.g. [1,12]). The mathematical background employed in those investigations may involve rather sophisticated techniques such as algebraic curves, functional analysis, matrix theory, etc. (see [16,7,8,2,18] and the references therein).

Inverse eigenvalue problems for band matrices have been actively investigated, e.g. see [7] and their references. The inverse spectral problem for a *periodic Jacobi matrix*, that is, a real symmetric matrix of the form

$$L_n = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 & b_n \\ b_1 & a_2 & b_2 & \dots & 0 & 0 \\ 0 & b_2 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_{n-1} \\ b_n & 0 & 0 & \dots & b_{n-1} & a_n \end{pmatrix}, \quad b_i > 0, \tag{1.1}$$

deserved the attention of researchers, see [8,9,2,19] and their references. These matrices appear in studies of the periodic Toda lattice, inverse eigenvalue problems for Sturm–Liouville equations and Hill's equation [7,9]. If $b_n = 0$, the matrices L_n of the form (1.1) reduce to tridiagonal symmetric matrices called the *Jacobi matrices*. The Jacobi matrices motivated intensive study as an useful tool in the investigation of orthogonal polynomials, in the theory of continued fractions, and in numerical analysis [20,6,5]. Namely, the inverse problems for Jacobi matrices have been an intensive topic of research since the seminal papers by Hochstadt and Hald [14,13] in the seventies of the last century.

In the present work, we study spectral properties of complex matrices of the form

$$J_n = \begin{pmatrix} c_1 & b_1 & 0 & \dots & 0 & \bar{b}_n \\ \bar{b}_1 & c_2 & b_2 & \dots & 0 & 0 \\ 0 & \bar{b}_2 & c_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_{n-1} & b_{n-1} \\ b_n & 0 & 0 & \dots & \bar{b}_{n-1} & a_n \end{pmatrix}, \tag{1.2}$$

where $b_1, \dots, b_{n-1}, b_n \in \mathbb{C}\backslash\mathbb{R}, c_1, \dots, c_{n-1} \in \mathbb{R}$, and $a_n \in \mathbb{C}$, and solve the direct problem for such matrices. (Here $\bar{z}$ means the complex conjugate of z.) The matrices of the form (1.2) constitute the class $\mathcal{J}_n$.

We also solve the inverse spectral problem for matrices from the subclass $\hat{\mathcal{J}}_n$ of the class $\mathcal{J}_n$. This subclass consists of the matrices of the form

$$\hat{J}_n = \begin{pmatrix} \hat{c}_1 & \hat{b}_1 & 0 & \dots & 0 & \bar{\hat{b}}_n \\ \hat{b}_1 & \hat{c}_2 & \hat{b}_2 & \dots & 0 & 0 \\ 0 & \hat{b}_2 & \hat{c}_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \hat{c}_{n-1} & \hat{b}_{n-1} \\ \hat{b}_n & 0 & 0 & \dots & \hat{b}_{n-1} & \hat{a}_n \end{pmatrix}, \tag{1.3}$$

where $\hat{b}_1, \dots, \hat{b}_{n-1}, \hat{c}_1, \dots, \hat{c}_{n-1} \in \mathbb{R}$, and $\hat{a}_n, \hat{b}_n \in \mathbb{C}$, $\hat{b}_n \neq 0$.

Note that in [3] (see also [4]), Arlinskii and Tsekhanovskii considered the matrices of the form (1.3) with $b_n = 0$, $a_n \in \mathbb{C}\backslash\mathbb{R}$, and solved the direct and inverse eigenvalue problems for those matrices. In [19], the direct and inverse spectral problems for the matrices the form (1.3) with $b_n \in \mathbb{R}\backslash\{0\}$ and $a_n \in \mathbb{R}$ (that is, for the matrices of the form (1.1)) were solved, and necessary and sufficient conditions for solvability of the inverse problem were found.

Recall that in [19], it was established that the necessary and sufficient conditions for the inverse spectral problem for the matrices of the form (1.1) to be solvable are