Linear Algebra and its Applications 439 (2013) 3561–3575

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On sum of powers of the Laplacian eigenvalues of graphs

Kinkar Ch. Das[a,*], Kexiang Xu[b], Muhuo Liu[c,d]

[a] Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea
[b] College of Science, Nanjing University of Aeronautics & Astronautics, Nanjing, PR China
[c] Department of Applied Mathematics, South China Agricultural University, Guangzhou, 510642, China
[d] Institute of Mathematics, School of Mathematical Science, Nanjing Normal University, Nanjing, 210046, China

ARTICLE INFO

Article history:
Received 7 June 2013
Accepted 23 September 2013
Available online 9 October 2013
Submitted by R. Brualdi

MSC:
05C50

Keywords:
Graph
Laplacian Matrix
Laplacian eigenvalues
Maximum degree
Independence number
Clique number
Number of spanning trees

ABSTRACT

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The Laplacian matrix of G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of its vertex degrees and $A(G)$ is the adjacency matrix. Let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-1} \geqslant \mu_n = 0$ be the Laplacian eigenvalues of G. For a graph G and a real number $\beta \neq 0$, the graph invariant $S_\beta(G)$ is the sum of the β-th power of the non-zero Laplacian eigenvalues of G, that is,

$$S_\beta(G) = \sum_{i=1}^{n-1} \mu_i^\beta.$$

In this paper, we obtain some lower and upper bounds on $S_\beta(G)$ for G in terms of n, the number of edges m, maximum degree Δ_1, clique number ω, independence number α and the number of spanning trees t. Moreover, we present some Nordhaus–Gaddum-type results for $S_\beta(G)$ of G.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, $|E(G)| = m$. Let d_i be the degree of the vertex v_i for $i = 1, 2, \ldots, n$. The maximum degree is denoted by Δ_1, the second maximum degree by Δ_2 and the minimum by δ. Let t and $\bar{t}$ be the number

* Corresponding author.
E-mail addresses: kinkardas2003@googlemail.com (K.Ch. Das), kexxu1221@126.com (K. Xu), liumuhuo@163.com (M. Liu).

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.09.036

of spanning trees in G and $\overline{G}$, respectively. Let $\mathbf{A}(G)$ be the $(0,1)$-adjacency matrix of G and $\mathbf{D}(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$. This matrix has nonnegative eigenvalues $n \geqslant \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n = 0$. Denote by $Spec(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ the spectrum of $\mathbf{L}(G)$, i.e., the Laplacian spectrum of G. When more than one graph is under consideration, then we write $\mu_i(G)$ instead of μ_i. It is known that $\mu_n = 0$ and the multiplicity of 0 is equal to the number of connected components of G.

As well known [16], a graph of order n has

$$t = t(G) = \frac{1}{n}\prod_{i=1}^{n-1}\mu_i \tag{1}$$

spanning trees and

$$\sum_{i=1}^{n}\mu_i = 2m. \tag{2}$$

For a graph G and a real number $\beta \neq 0$, the graph invariant $S_\beta(G)$ is the sum of the β-th power of the non-zero Laplacian eigenvalues of G, that is,

$$S_\beta(G) = \sum_{i=1}^{n-1}\mu_i^\beta.$$

For its basic properties, including various upper and lower bounds, see [13,20,23].

In 2008, Liu and Liu [14] considered a new Laplacian-spectrum-based graph invariant

$$LEL = LEL(G) = \sum_{k=1}^{n-1}\sqrt{\mu_k} \tag{3}$$

and named it *Laplacian-energy-like invariant* (or *LEL* for short). *Laplacian-energy-like invariant* is a special case of the graph invariant $S_\beta(G)$ when $\beta = 1/2$. The motivation for introducing *LEL* was in its analogy to the earlier much studied graph energy [7,11] and Laplacian energy [8]. For details on *LEL* see the review [15], the recent papers [3,4,9,19,22], and the references cited therein.

A maximum clique of a graph G is a *clique* (i.e., complete subgraph) of maximum possible size for G. The size of the maximum clique is known as a graph's *clique number* and it is denoted by ω. A *complete split graph* $CS(n,\omega)$, $\omega \leqslant n$, is a graph on n vertices consisting of a clique on ω vertices and a stable set on the remaining $n-\omega$ vertices in which each vertex of the clique is adjacent to each vertex of the stable set. Given a graph G, a subset S of $V(G)$ is called an *independent set* of G if $G[S]$, an induced subgraph by S, is a graph with $|S|$ isolated vertices. The *independence number* of G is denoted by α and is defined to be the number of vertices in the largest independent set of G. The Laplacian spectrum of complete split graph $CS(n, n-\alpha)$ $(1 \leqslant \alpha \leqslant n-1)$ [4] is

$$(\underbrace{n, n, \ldots, n}_{n-\alpha}, \underbrace{n-\alpha, n-\alpha, \ldots, n-\alpha}_{\alpha-1}, 0). \tag{4}$$

As usual, K_n, P_n, and $K_{1,n-1}$, denote, respectively, the complete graph, the path, and the star on n vertices.

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we obtain some lower and upper bounds on $S_\beta(G)$ for G in terms of n, the number of edges m, maximum degree Δ_1, clique number ω, independence number α and the number of spanning trees t. In Section 4, we present some Nordhaus–Gaddum-type results for $S_\beta(G)$ of graphs.

2. Preliminaries

In this section, we shall list some previously known results that will be needed in the next two sections. Arithmetic mean and geometric mean are related in the following lemma: