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An operator inequality and its consequences

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ABSTRACT

Let f be a continuous convex function on an interval J, let A, B, C, D be self-adjoint operators acting on a Hilbert space with spectra contained in J such that $A + D = B + C$ and $A \leqslant m \leqslant B, C \leqslant M \leqslant D$ for two real numbers $m < M$, and let Φ be a unital positive linear map on $\mathbb{B}(\mathscr{H})$. We prove the inequality

$$f(\Phi(B)) + f(\Phi(C)) \leqslant \Phi(f(A)) + \Phi(f(D))$$

and apply it to obtain several inequalities such as the Jensen–Mercer operator inequality and the Petrović operator inequality.

1. Introduction and preliminaries

Throughout the paper $\mathbb{B}(\mathscr{H})$ stands for the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ and I denotes the identity operator. The real subspace of $\mathbb{B}(\mathscr{H})$ consisting of all self-adjoint operators on $\mathscr{H}$ is denoted by $\mathbb{B}(\mathscr{H})_h$. We extensively use the continuous functional calculus

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for self-adjoint operators, e.g., see [6]. An operator A is said to be positive (denoted by $A \geqslant 0$) if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. If, in addition, A is invertible, then it is called strictly positive (denoted by $A > 0$). By $A \geqslant B$ we mean that $A - B$ is positive, while $A > B$ means that $A - B$ is strictly positive. A map Φ on $\mathbb{B}(\mathscr{H})$ is said to be positive if $\Phi(A) \geqslant 0$ for each $A \geqslant 0$ and is called unital if $\Phi(I) = I$.

Let f be a continuous real valued function defined on an interval J. The function f is called operator monotone if $A \geqslant B$ implies $f(A) \geqslant f(B)$ for all $A, B \in \mathbb{B}(\mathscr{H})_h$ with spectra in J. A function f is called operator convex on J if $f(\lambda A + (1-\lambda)B) \leqslant \lambda f(A) + (1-\lambda)f(B)$ or all $A, B \in \mathbb{B}(\mathscr{H})_h$ with spectra in J and all $\lambda \in [0, 1]$. If the function f is operator convex, then the so-called Jensen operator inequality $f(\Phi(A)) \leqslant \Phi(f(A))$ holds for any unital positive linear map Φ on $\mathbb{B}(\mathscr{H})$ and any $A \in \mathbb{B}(\mathscr{H})_h$ with spectrum contained in J. Many other versions of the Jensen operator inequality can be found in [7]. Among them, the Jensen–Mercer operator inequality [10] reads as follows:

$$f\left(M + m - \sum_{i=1}^{n} \Phi_i(A_i)\right) \leqslant f(M) + f(m) - \sum_{i=1}^{n} \Phi_i(f(A_i)),$$

where f is a continuous convex function on an interval $[m, M]$, $\Phi_1, \ldots, \Phi_n$ are positive linear maps on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^{n} \Phi_i(I) = I$ and $A_1, \ldots, A_n \in \mathbb{B}(\mathscr{H})_h$ with spectra contained in $[m, M]$, see also [8].

Another interesting inequality is the Petrović inequality (see e.g. [12, page 152]). It states that if $f : [0, \infty) \to \mathbb{R}$ is a continuous convex function and $A_1, \ldots, A_n$ are positive operators such that $\sum_{i=1}^{n} A_i = MI$ for some scalar $M > 0$, then

$$\sum_{i=1}^{n} f(A_i) \leqslant f\left(\sum_{i=1}^{n} A_i\right) + (n-1)f(0).$$

If $f : [0, \infty) \to [0, \infty)$ is a convex function and $f(0) \leqslant 0$, then

$$f(a) + f(b) \leqslant f(a+b) \tag{1}$$

for all scalars $a, b \geqslant 0$. However, if the scalars a, b are replaced by two positive operators, this inequality may not hold. There have been many interesting works devoted to obtain norm or operator extensions of inequality (1) and other subadditivity inequalities. Ando and Zhan [1] showed that if f is a nonnegative operator monotone function on the interval $[0, \infty)$, then

$$|||f(A+B)||| \leqslant |||f(A) + f(B)||| \tag{2}$$

for all unitarily invariant norms $|||\cdot|||$ and all matrices $A, B \geqslant 0$. Bourin and Uchiyama [5] extended Ando—Zhan's result by showing that if $A, B \geqslant 0$ and f is a non-negative concave function on $[0, \infty)$, then (2) holds for all unitarily norms. Also Aujla and Bourin [3] showed that if $A, B \geqslant 0$ and $f : [0, \infty) \to [0, \infty)$ is a monotone concave function, then there exist unitaries U, V such that

$$f(A+B) \leqslant Uf(A)U^* + Vf(B)V^*.$$

The reader is referred to [4,13,9,14,11,2] and references therein for recent information on the subject.

In their study of Hadamard's inequalities for co-ordinated convex functions, Hwang, Tseng and Yang proved that if $f : [a, b] \to \mathbb{R}$ is a convex function and $a \leqslant y_1 \leqslant x_1 \leqslant x_2 \leqslant y_2 \leqslant b$ and $x_1 + x_2 = y_1 + y_2$, then

$$f(x_1) + f(x_2) \leqslant f(y_1) + f(y_2).$$

In this paper, we extend this inequality to operators acting on a Hilbert space and apply it to obtain a series of operator inequalities including the Jensen–Mercer operator inequality and the Petrović operator inequality.