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Subconstituents of orthogonal graphs of odd characteristic – continued

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ABSTRACT

The subconstituents of the orthogonal graph $O(2\nu+\delta,q)$, where $\nu \geqslant 2$ and $\delta \in \{1,2\}$, over a finite field of odd characteristic are shown to be quasi-strongly regular. Furthermore, the first subconstituent is shown to be co-edge regular, and when $\nu \geqslant 3$ its automorphism group is determined. The second subconstituent is shown to be edge regular, and when $\nu \geqslant 2$ its automorphism group is determined. Their parameters and chromatic numbers are also determined.

1. Introduction

We study the orthogonal graphs over finite fields of odd characteristic. These graphs are known to be strongly regular [7]. In [8], we initiated a study of such graphs; in this paper, we continue their study by considering their subconstituents. We show that both subconstituents are quasi-strongly regular, and we compute their parameters and chromatic numbers.

The orthogonal graphs of odd characteristic were first mentioned in [9] as one of the families of strongly regular graphs constructed by Chevally groups. In [7], they were reconstructed and studied by the geometry of orthogonal groups over finite fields, and some of their properties were described. Their construction proceeds as follows.

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Let $\mathbb{F}_q$ be a finite field of odd characteristic and $\mathbb{F}_q^{(n)}$ be the n-dimensional row vector space over $\mathbb{F}_q$. For any positive integer ν and $\delta \in \{0,1,2\}$, let

$$S_{2\nu+\delta} = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & \Delta \end{pmatrix},$$

where $I^{(\nu)}$ is the $\nu \times \nu$ identity matrix, and

$$\Delta = \begin{cases} \emptyset & \text{if } \delta = 0, \\ 1 & \text{if } \delta = 1, \\ \begin{pmatrix} 1 & \\ & -z \end{pmatrix} & \text{if } \delta = 2, \end{cases}$$

where z is a fixed non-square element of $\mathbb{F}_q^*$ and $\emptyset$ is always omitted if $\delta = 0$. For vectors $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{F}_q^{(2\nu+\delta)}$, let $[\alpha_1, \alpha_2, \ldots, \alpha_s]$ denote their span. Given a nonzero vector $\alpha \in \mathbb{F}_q^{(2\nu+\delta)}$, we say that α is *isotropic* (with respect to $S_{2\nu+\delta}$) whenever $\alpha S_{2\nu+\delta}{}^t\alpha = 0$. We say that the vector space $[\alpha]$ is *isotropic* whenever α is.

The *orthogonal graph with respect to* $S_{2\nu+\delta}$, denoted by $O(2\nu+\delta,q)$, is the graph with the set of all the 1-dimensional isotropic spaces as the vertex set $V(O(2\nu+\delta,q))$ and with the adjacency $[\alpha] \sim [\beta]$ defined by $\alpha S_{2\nu+\delta}{}^t\beta \neq 0$.

Recall that a connected regular graph G with degree k on v vertices is *edge-regular* if any two adjacent vertices have λ common neighbors; is *co-edge regular* if any two nonadjacent vertices have μ common neighbors. Moreover, G is *strongly regular* with parameter (v,k,λ,μ) if any two adjacent vertices have λ common neighbors and any two nonadjacent vertices have μ common neighbors. It was shown in [7] that when $\nu = 1$, $O(2\nu+\delta,q)$ is a complete graph of $q^\delta + 1$ vertices and when $\nu \geqslant 2$, $O(2\nu+\delta,q)$ is a strongly regular graph with parameters

$$\left((q^\nu - 1)(q^{\nu+\delta-1} + 1)(q-1)^{-1}, q^{2\nu+\delta-2}, \lambda, \mu\right), \tag{1.1}$$

where $\lambda = q^{2\nu+\delta-2} - q^{2\nu+\delta-3} - q^{\nu-1} + q^{\nu+\delta-2}$ and $\mu = q^{2\nu+\delta-2} - q^{2\nu+\delta-3}$. The readers are referred to [1,4,11] for further discussion of strongly regular graphs and to [7,13] for further discussion of the orthogonal group and the orthogonal graph over a finite field of odd characteristic.

To describe our main results we require two generalizations of strongly regular graphs given by W. Golightly, W. Haynworth and D.G. Sarvate [5]. Let G be a connected regular graph of degree k on v vertices and $c_1, c_2, \ldots, c_d$ be distinct integers. G is said to be *quasi-strongly regular* with parameters $(v,k,\lambda; c_1, c_2, \ldots, c_d)$ if any two adjacent vertices have λ common neighbors, and any two nonadjacent vertices have c_i common neighbors for some $i (1 \leqslant i \leqslant d)$. Also, G is said to be *quasi-strongly regular* with parameters $(v,k,c_1,c_2,\ldots,c_d;\mu)$ if any two nonadjacent vertices have μ common neighbors, and any two adjacent vertices have c_i common neighbors for some $i (1 \leqslant i \leqslant d)$.

Note that the diameter of $O(2\nu+\delta,q)$ is 2 when $\nu \geqslant 2$. In the rest of this paper, we assume $\nu \geqslant 2$. For any $[\alpha] \in V(O(2\nu+\delta,q))$, the *i-th subconstituent* $\Gamma_i([\alpha])$ with respect to $[\alpha]$ is the induced subgraph of $O(2\nu+\delta,q)$ with vertices at distance i from $[\alpha]$, where $i = 1, 2$. The subconstituents of $O(2\nu,q)$ were studied in [8]. The first subconstituent of $O(2\nu,q)$ is strongly regular, and the second subconstituent is strongly regular when $\nu = 2$, and quasi-strongly regular when $\nu \geqslant 3$. Their full automorphism groups were also determined there. In the present paper, we study the subconstituents of $O(2\nu+\delta,q)$ for $\delta = 1$ and $\delta = 2$. Denote by $e_i \in \mathbb{F}_q^{(n)}$ the row vector whose i-th component is 1 and all other components are $0, 1 \leqslant i \leqslant n$. The orthogonal group $O_{2\nu+\delta}(\mathbb{F}_q)$ acts as a transitive group of automorphisms on the orthogonal graph $O(2\nu+\delta,q)$ [7]. Thus the isomorphism class of each subconstituent does not depend upon vertex. We refer to any graph in such an isomorphism class as the corresponding subconstituent. Hence to study the subconstituents of $O(2\nu+\delta,q)$, it suffices to consider $\Gamma_i([e_1])$, $i = 1$ or 2, which is denoted by Γ_i for simplicity.

The paper is organized as follows. In Section 2, we study the action of the orthogonal group $O_{2\nu+\delta}(\mathbb{F}_q)$ on the orthogonal graph $O(2\nu+\delta,q)$ for $\delta = 1$ and $\delta = 2$ in preparation of later sections. In Section 3 we shall show that Γ_1 of $O(2\nu+1,q)$ is quasi-strongly regular and Γ_1 of $O(2\nu+2,q)$ is strongly regular. In Section 4, we shall show that when $\nu = 2$, Γ_2 of $O(2\nu+\delta,q)$, $\delta \in \{1,2\}$, is strongly regular; and when $\nu \geqslant 3$, it is quasi-strongly regular. In particular, Γ_2 of $O(2\nu+1,q)$ is