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Nonlinear perturbations of linear equations in $\mathbb{R}^n$

W. Phillip Forrester, Edgar E. Galindo, Adolfo J. Rumbos *

Department of Mathematics, Pomona College, Claremont, CA 91711, United States

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ABSTRACT

The main goal of this note is to prove an existence theorem for the semilinear equation

$$Ax = \lambda x + g(x), \quad x \in \mathbb{R}^n, \tag{0.1}$$

for the case in which A is a symmetric, $n \times n$ matrix with real entries, the vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ is a bounded continuous function, and λ is a simple eigenvalue of the matrix A. In particular, let u denote an eigenvector of A corresponding to λ with $\|u\| = 1$, and assume that the real valued functions

$$p_+(\omega, u) = \lim_{r\to\infty} \langle g(r\omega), u\rangle, \quad \text{for all } \omega \in S_1,$$

and

$$p_-(\omega, u) = \lim_{r\to-\infty} \langle g(r\omega), u\rangle, \quad \text{for all } \omega \in S_1,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$, are defined on the unit sphere $S_1 = \partial B_1(0)$, where $B_1(0)$ denotes the open unit ball in $\mathbb{R}^n$.

The main theorem in this work states that if $p_+(\cdot, u)$ and $p_-(\cdot, u)$ are continuous in a neighborhood of the point u in S_1, and

$$p_+(u, u) < 0 < p_-(u, u),$$

then the equation in (0.1) has at least one solution.

* Corresponding author.
E-mail address: arumbos@pomona.edu (A.J. Rumbos).

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1. Introduction

Let A denote a symmetric, $n \times n$ matrix with real entries. It is well known that the eigenvalues of A are real and that there exists a basis for $\mathbb{R}^n$ of eigenvectors of A that are mutually orthogonal and that have Euclidean norm equal to 1; that is, there exists an orthonormal basis for $\mathbb{R}^n$ made up of eigenvectors of A. We denote the set of eigenvalues of A by $\sigma(A)$ and call it the spectrum of A.

It is also well known that, if $\lambda \notin \sigma(A)$, then the linear equation

$$Ax = \lambda x + b \tag{1.1}$$

has a unique solution for any $b \in \mathbb{R}^n$. On the other hand, if $\lambda \in \sigma(A)$, then the equation in (1.1) is solvable if and only if the vector b is orthogonal to the vectors in the eigenspace, E_λ, corresponding to λ; in other words, the equation in (1.1) is solvable if and only if

$$\langle b, w\rangle = 0, \quad \text{for all } w \in E_\lambda, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. However, in this case the solution is not unique.

The preceding is a special case of the Fredholm Alternative for Matrices (in the case of symmetric matrices) and the expression in (1.2) can be thought of as a solvability condition for the equation in (1.1). In this note we would like to discuss some extensions of the Fredholm Alternative to the case of the semilinear equation

$$Ax = \lambda x + g(x), \tag{1.3}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field.

It can be shown, using the Brouwer Fixed Point Theorem, for instance, that if $\lambda \notin \sigma(A)$ and g satisfies the growth condition

$$\lim_{\|x\|\to\infty} \frac{\|g(x)\|}{\|x\|} = 0,$$

then the semilinear equation in (1.3) has at least one solution in $\mathbb{R}^n$ (see Theorem 3.3 in Section 3.2 of this note). Thus, the first option in the Fredholm Alternative is obtained in the semilinear case, except for the assertion of uniqueness. This is known as the nonresonance case.

The main goal of this paper is to show that the sufficiency part of the second option in the Fredholm Alternative holds true for a special case of the semilinear equation in (1.3); specifically, we will consider the case in which the vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ is bounded and $\lambda \in \sigma(A)$ is a simple eigenvalue of A; that is, $\dim(E_\lambda) = 1$.

Let u be a unit vector in E_λ. Assume that real valued functions

$$p_+(\omega, u) = \lim_{r\to\infty} \langle g(r\omega), u\rangle, \quad \text{for all } \omega \in S_1, \tag{1.4}$$

and

$$p_-(\omega, u) = \lim_{r\to-\infty} \langle g(r\omega), u\rangle, \quad \text{for all } \omega \in S_1 \tag{1.5}$$

can be defined on the unit sphere $S_1 = \partial B_1(0)$, where $B_1(0)$ denotes the open unit ball in $\mathbb{R}^n$. We will prove the following result:

Theorem 1.1. *Suppose that $\lambda \in \sigma(A)$ and that λ is a simple eigenvalue of A. Let u denote an eigenvector of A corresponding to λ with $\|u\| = 1$. Assume that the functions $p_+(\cdot, u) : S_1 \to \mathbb{R}$ and $p_-(\cdot, u) : S_1 \to \mathbb{R}$ defined in (1.4) and (1.5), respectively, are continuous in a neighborhood of the point u in S_1, and that*

$$p_+(u, u) < 0 < p_-(u, u). \tag{1.6}$$

Then, the equation in (1.3) has at least one solution.