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Generalizations of operator Shannon inequality based on Tsallis and Rényi relative entropies

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ABSTRACT

Recently, we obtained relations among relative operator entropies of sequences. For example, for relative operator entropy $S(\mathbb{A}|\mathbb{B})$, Rényi relative operator entropy $I_t(\mathbb{A}|\mathbb{B})$ and Tsallis relative operator entropy $T_t(\mathbb{A}|\mathbb{B})$ of sequences of strictly positive operators $\mathbb{A} = (A_1, \dots, A_n)$ and $\mathbb{B} = (B_1, \dots, B_n)$ such that $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$,

$$S(\mathbb{A}|\mathbb{B}) \leqslant I_t(\mathbb{A}|\mathbb{B}) \leqslant T_t(\mathbb{A}|\mathbb{B}) \leqslant 0$$

holds for $0 < t < 1$. This is an extension of operator version of Shannon inequality (briefly, operator Shannon inequality) discussed by Furuta and Yanagi–Kuriyama–Furuichi. In this paper, we shall obtain two generalizations of this inequality by considering generalizations of relative operator entropies of sequences.

1. Introduction

In this paper, an operator means a bounded linear operator on a Hilbert space $\mathcal{H}$. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in \mathcal{H}$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

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For two strictly positive operators A and B, the weighted geometric mean is defined by $A \natural_t B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}}$ for $0 \leqslant t \leqslant 1$. We can treat $A \natural_t B$ as a path from A to B. For $A, B > 0$ and $t \in \mathbb{R}$, Furuta [1] introduced

$$S_t(A|B) \equiv A^{\frac{1}{2}}\big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\big)^t \log\big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\big)A^{\frac{1}{2}}.$$

$S_t(A|B)$ can be considered as a tangent at t of $A \natural_t B$. If $t = 0$, then

$$S(A|B) \equiv S_0(A|B) = A^{\frac{1}{2}} \log\big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\big)A^{\frac{1}{2}}.$$

$S(A|B)$ is called relative operator entropy, so we call $S_t(A|B)$ generalized relative operator entropy.

Tsallis relative operator entropy is introduced by Yanagi, Kuriyama and Furuichi [5] as follows: For $A, B > 0$ and $0 < t \leqslant 1$,

$$T_t(A|B) \equiv \frac{A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}} - A}{t} = \frac{A \natural_t B - A}{t}.$$

We remark that

$$T_0(A|B) \equiv \lim_{t \to +0} T_t(A|B) = S(A|B)$$

since $\lim_{t \to +0} \frac{x^t - 1}{t} = \log x$ for $x > 0$, and also the definition of $T_t(A|B)$ can be extended for $t \in \mathbb{R}$.

Let $\mathbb{A} = (A_1, \ldots, A_n)$ and $\mathbb{B} = (B_1, \ldots, B_n)$ be sequences of strictly positive operators. We define relative operator entropy $S(\mathbb{A}|\mathbb{B})$, generalized relative operator entropy $S_t(\mathbb{A}|\mathbb{B})$, Tsallis relative operator entropy $T_t(\mathbb{A}|\mathbb{B})$ and Rényi relative operator entropy $I_t(\mathbb{A}|\mathbb{B})$ of two sequences $\mathbb{A}$ and $\mathbb{B}$ as follows: For $0 \leqslant t \leqslant 1$,

$$S(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^{n} S(A_i|B_i), \qquad S_t(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^{n} S_t(A_i|B_i),$$

$$T_t(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^{n} T_t(A_i|B_i) \quad \text{and}$$

$$I_t(\mathbb{A}|\mathbb{B}) \equiv \frac{1}{t} \log \sum_{i=1}^{n} A_i \natural_t B_i \quad (\text{if } t \neq 0).$$

In this paper, we assume $\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i = I$. We remark that

$$I_0(\mathbb{A}|\mathbb{B}) \equiv \lim_{t \to +0} I_t(\mathbb{A}|\mathbb{B}) = S(\mathbb{A}|\mathbb{B})$$

follows from (1.3) stated below.

On the other hand, under the above assumption, Furuta [1] obtained the operator version of Shannon inequality (briefly, operator Shannon inequality).

$$S(\mathbb{A}|\mathbb{B}) \leqslant 0. \tag{1.1}$$

Yanagi, Kuriyama and Furuichi [5] obtained a generalization of (1.1) by using Tsallis relative operator entropy of sequences.

$$T_t(\mathbb{A}|\mathbb{B}) \leqslant 0 \quad \text{for } 0 < t \leqslant 1. \tag{1.2}$$

Recently, as relations among relative operator entropies of sequences, we obtained the following inequalities including (1.1) and (1.2).