Linear Algebra and its Applications 437 (2012) 2359–2365

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

An extension of the Löwner–Heinz inequality

Mohammad Sal Moslehian *, Hamed Najafi

Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran

ARTICLE INFO

Article history:
Received 3 April 2012
Accepted 16 May 2012
Available online 6 July 2012

Submitted by H. Schneider

AMS classification:
Primary: 47A63
Secondary: 47B10
47A30

Keywords:
Löwner–Heinz inequality
Positive operator
Operator monotone function

ABSTRACT

We extend the celebrated Löwner–Heinz inequality by showing that if A, B are Hilbert space operators such that $A > B \geqslant 0$, then

$$A^r - B^r \geqslant ||A||^r - \left(||A|| - \frac{1}{||(A-B)^{-1}||}\right)^r > 0$$

for each $0 < r \leqslant 1$. As an application we prove that

$$\log A - \log B \geqslant \log ||A|| - \log\left(||A|| - \frac{1}{||(A-B)^{-1}||}\right) > 0.$$

1. Introduction

Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$ equipped with the operator norm $\|\cdot\|$. There are three types of ordering on the real space of all self-adjoint operators as follows. Let $A, B \in \mathbb{B}(\mathscr{H})$ be self-adjoint. Then

(1) $A \geqslant B$ if $\langle Ax, x\rangle \geqslant \langle Bx, x\rangle$.
(2) $A \succ B$ if $\langle Ax, x\rangle > \langle Bx, x\rangle$ holds for all non-zero elements $x \in \mathscr{H}$.
(3) $A > B$ if $A \geqslant B$ and $A - B$ is invertible.

Clearly (3)$\Rightarrow$ (2)$\Rightarrow$ (1) but the reverse implications are not valid in general. For instance, if A is the diagonal operator $(1, 1/2, 1/3, \ldots)$ on ℓ^2, then $A \succ 0$ but $A \not> 0$. Of course, in the case where H is of

* Corresponding author.
E-mail addresses: moslehian@ferdowsi.um.ac.ir, moslehian@member.ams.org (M.S. Moslehian), hamednajafi20@gmail.com (H. Najafi).

0024-3795/$ - see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.laa.2012.05.027

finite dimension, (2) and (3) are equivalent. A continuous real valued function f defined on an interval J is called operator monotone if $A \geqslant B$ implies that $f(A) \geqslant f(B)$ for all self-adjoint operators A, B with spectra in J. The Löwner–Heinz inequality says that, $f(x) = x^r$ $(0 < r \leqslant 1)$ is operator monotone on $[0, \infty)$. Löwner [10] proved the inequality for matrices. Heinz [8] proved it for positive operators acting on a Hilbert space of arbitrary dimension. Based on the C^*-algebra theory, Pedersen [11] gave a shorter proof of the inequality.

There exist several operator norm inequalities each of which is equivalent to the Löwner–Heinz inequality; see [7]. One of them is $\|A^r B^r\| \leq \|AB\|^r$, called the Cördes inequality in the literature, in which A and B are positive operators and $0 < r \leq 1$. A generalization of the Cördes inequality for operator monotone functions is given in [4]. It is shown in [1] that this norm inequality is related to the Finsler structure of the space of positive invertible elements.

Kwong [9] showed that if $A > B$ ($A \succ B$, resp.), then $A^r > B^r$ ($A^r \succ B^r$, resp.) for $0 < r \leqslant 1$. Uchiyama [12] showed that for every non-constant operator monotone function f on an interval J, $A \succ B$ implies $f(A) \succ f(B)$ for all self-adjoint operators A, B with spectra in J.

There are several extensions of the Löwner–Heinz inequality. The Furuta inequality [6], which states that if $A \geq B \geq 0$, then for $r \geq 0$, $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$ holds for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p + r$, is known as an exquisite extension of the Löwner–Heinz inequality; see the survey article [5] and references therein.

If f is an operator monotone function on $(-1, 1)$, then f can be represented as

$$f(t) = f(0) + f^{'}(0) \int_{-1}^{1} \frac{t}{1-\lambda t} d\mu(\lambda), \tag{1.1}$$

where μ is a positive measure on $(-1, 1)$. It is known that

$$t^r = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{t}{\lambda + t} \lambda^{r-1} d\lambda, \tag{1.2}$$

in which $0 < r < 1$, and

$$A^r = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{A}{\lambda + A} \lambda^{r-1} d\lambda, \tag{1.3}$$

where A is positive and $0 < r < 1$; see e.g. [3, Chapter V].

In this paper we extend the Löwner–Heinz inequality by showing that if $A, B \in \mathbb{B}(\mathscr{H})$ such that $A > B \geqslant 0$, then

$$A^r - B^r \geqslant ||A||^r - \left(||A|| - \frac{1}{||(A-B)^{-1}||}\right)^r > 0$$

for each $0 < r \leqslant 1$. As an application we prove that

$$\log A - \log B \geqslant \log ||A|| - \log\left(||A|| - \frac{1}{||(A-B)^{-1}||}\right) > 0.$$

2. The results

We start our work with the following useful lemma.

Lemma 2.1. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be invertible positive operators such that $A - B \geqslant m > 0$. Then*

$$B^{-1} - A^{-1} \geqslant \frac{m}{(||A|| - m)\, ||A||}\,. \tag{2.1}$$