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Maximum norms of graphs and matrices, and their complements

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ABSTRACT

Given a graph G, let $\|G\|_*$ denote the trace norm of its adjacency matrix, also known as the *energy* of G. The main result of this paper states that if G is a graph of order n, then

$$\|G\|_* + \left\|\overline{G}\right\|_* \leqslant (n-1)\left(1+\sqrt{n}\right),$$

where $\overline{G}$ is the complement of G. Equality is possible if and only if G is a strongly regular graph with parameters $(n, (n-1)/2, (n-5)/4, (n-1)/4)$, known also as a conference graph.

In fact, the above problem is stated and solved in a more general setup - for nonnegative matrices with bounded entries. In particular, this study exhibits analytical matrix functions attaining maxima on matrices with rigid and complex combinatorial structure.

In the last section the same questions are studied for Ky Fan norms. Possible directions for further research are outlined, as it turns out that the above problems are just a tip of a larger multidimensional research area.

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1. Introduction and main results

In this paper we study the maxima of certain norms of nonnegative matrices with bounded entries. We shall focus first on the trace norm $\|A\|_*$ of a matrix A, which is just the sum of its singular values. The trace norm of the adjacency matrix of graphs has been intensively studied recently under the

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name *graph energy*. This research has been initiated by Gutman in [3]; the reader is referred to [6] for a comprehensive recent survey and references.

One of the most intriguing problems in this area is to determine which graphs with given number of vertices have maximal energy. A cornerstone result of Koolen and Moulton [5] shows that if G is a graph of order n, then the trace norm of its adjacency matrix $\|G\|_*$ satisfies the inequality

$$\|G\|_* \leqslant \left(1+\sqrt{n}\right)\frac{n}{2}, \tag{1}$$

with equality holding precisely when G is a strongly regular graph with parameters

$$\left(n, \left(n+\sqrt{n}\right)/2, \left(n+2\sqrt{n}\right)/4, \left(n+2\sqrt{n}\right)/4\right).$$

Note that G can be characterized also as a regular graph of degree $\left(n+\sqrt{n}\right)/2$, whose singular values, other than the largest one, are equal to $\sqrt{n}/2$. Graphs with such properties can exist only if n is an even square. It is easy to see that these graphs are related to Hadamard matrices, and indeed such relations have been outlined in [4].

As it turns out, if for a graph G equality holds in (1), then its complement $\overline{G}$ is a strongly regular graph which is quite similar but not isomorphic to G, and therefore with $\left\|\overline{G}\right\|_* < \left(1+\sqrt{n}\right)n/2$. This observation led Gutman and Zhou [10] to the following natural question:

What is the maximum $\mathcal{E}(n)$ of the sum $\|G\|_* + \left\|\overline{G}\right\|_*$, where G is a graph of order n?

Questions of this type are the subject matter of this paper. To begin with, note that the Paley graphs, and more generally conference graphs (see below), provide a lower bound

$$\mathcal{E}(n) \geqslant (n-1)\left(1+\sqrt{n}\right), \tag{2}$$

which closely complements the upper bound

$$\mathcal{E}(n) \leqslant \sqrt{2}n + (n-1)\sqrt{n-1}$$

proved by Gutman and Zhou in [10]. Although the latter inequality can be improved by a more careful proof (see Theorem 5 below), a gap still remains between the upper and lower bounds on $\mathcal{E}(n)$. In this note we shall close this gap and show that inequality (2) is in fact an equality. This advancement comes at a price of a rather long proof, involving some new analytical and combinatorial techniques based on Weyl's inequalities for sums of Hermitian matrices.

Moreover, we prove our bounds for nonnegative matrices that are more general than adjacency matrices of graphs, and yet the adjacency matrices of conference graphs provide the cases of equality. It is somewhat surprising that purely analytical matrix functions attain their maxima on matrices with a rigid and sophisticated combinatorial structure.

It is also natural to study similar problems for matrix norms different from the trace norm, like the Ky Fan or the Schatten norms. Other parameters can be changed independently and we thus arrive at a whole grid of extremal problems most of which are open and seem rather difficult. Results of this types and directions for further research are outlined in Section 3.

For reader's sake we start with our graph-theoretic result first.

Theorem 1. *If G is a graph of order $n \geqslant 7$, then*

$$\|G\|_* + \left\|\overline{G}\right\|_* \leqslant (n-1)\left(1+\sqrt{n}\right), \tag{3}$$

with equality holding if and only if G is a conference graph.

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