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The finite-step realizability of the joint spectral radius of a pair of $d \times d$ matrices one of which being rank-one ☆

Xiongping Dai [a], Yu Huang [b], Jun Liu [c], Mingqing Xiao [c,*]

[a] Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China
[b] Department of Mathematics, Zhongshan (Sun Yat-Sen) University, Guangzhou 510275, People's Republic of China
[c] Department of Mathematics, Southern Illinois University, Carbondale, IL 62901-4408, USA

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ABSTRACT

We study the finite-step realizability of the joint/generalized spectral radius of a pair of real square matrices S_1 and S_2, one of which has rank 1, where $2 \leq d < +\infty$. Let $\rho(A)$ denote the spectral radius of a square matrix A. Then we prove that there always exists a finite-length word $(i_1^*, \ldots, i_m^*) \in \{1, 2\}^m$, for some finite $m \geq 1$, such that

$$\sqrt[m]{\rho(\mathrm{S}_{i_1^*} \cdots \mathrm{S}_{i_m^*})} = \sup_{n \geq 1} \left\{ \max_{(i_1, \ldots, i_n) \in \{1,2\}^n} \sqrt[n]{\rho(\mathrm{S}_{i_1} \cdots \mathrm{S}_{i_n})} \right\}.$$

In other words, there holds the spectral finiteness property for $\{\mathrm{S}_1, \mathrm{S}_2\}$. Explicit formula for computation of the joint spectral radius is derived. This implies that the stability of the switched system induced by $\{\mathrm{S}_1, \mathrm{S}_2\}$ is algorithmically decidable in this case.

1. Introduction

Let $\boldsymbol{S} = \{\mathrm{S}_1, \ldots, \mathrm{S}_K\} \subset \mathbb{R}^{d \times d}$ be an arbitrary finite set of real d-by-d matrices and $\|\cdot\|$ a matrix norm on the space $\mathbb{R}^{d \times d}$ of all real $d \times d$ matrices, where $2 \leq d < +\infty$ and $K \geq 2$. To capture the maximal growth rate of the trajectories generated by random products of matrices $\mathrm{S}_1, \ldots, \mathrm{S}_K$ in $\boldsymbol{S}$, in 1960 [38] Rota and Strang introduced an important concept – *joint spectral radius of* $\boldsymbol{S}$ – by

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* Corresponding author.
E-mail addresses: xpdai@nju.edu.cn (X. Dai), stshyu@mail.sysu.edu.cn (Y. Huang), jliu@math.siu.edu (J. Liu), mxiao@math.siu.edu (M. Xiao).

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$$\hat{\rho}(\boldsymbol{S}) = \lim_{n\to+\infty}\left\{\max_{(i_1,\ldots,i_n)\in\mathbb{K}^n}\sqrt[n]{\|\mathrm{S}_{i_1}\cdots\mathrm{S}_{i_n}\|}\right\}.$$

Here and in the future

$$\mathbb{K}^n := \overbrace{\{1,\ldots,K\}\times\cdots\times\{1,\ldots,K\}}^{n\text{-times}}$$

stands for the set of all words $(i_1,\ldots,i_n)$ of finite-length n, composed of the letters $1,\ldots,K$, for any integer $n \geq 1$. Let

$$\Sigma_K^+ = \{i_{(\cdot)}\colon \mathbb{N}\to\mathbb{K}\}, \quad \text{where } \mathbb{N} = \{1,2,\ldots\},$$

be the set of all the one-sided infinite sequences (also called switching signals of $\boldsymbol{S}$). We write $i_{(\cdot)}$ as $i.$ for simplicity. Then we see, from Barabanov [1] for example, that $\hat{\rho}(\boldsymbol{S}) < 1$ if and only if

$$\|\mathrm{S}_{i_1}\cdots\mathrm{S}_{i_n}\| \to 0 \quad \text{as } n\to+\infty \qquad \forall i. \in \Sigma_K^+.$$

In other words, $\hat{\rho}(\boldsymbol{S}) < 1$ if and only if the linear switched dynamical system, also written as $\boldsymbol{S}$,

$$x_n = x_0\cdot\mathrm{S}_{i_1}\cdots\mathrm{S}_{i_n}, \quad x_0\in\mathbb{R}^d,\, n\geq 1, \text{ and } i.\in\Sigma_K^+,$$

is absolutely asymptotically stable, where the initial state $x_0 \in \mathbb{R}^d$ is an arbitrary given d-dimensional row vector. In fact, from [14] there follows

$$\hat{\rho}(\boldsymbol{S}) = \max_{i.\in\Sigma_K^+}\left\{\limsup_{n\to+\infty}\sqrt[n]{\|\mathrm{S}_{i_1}\cdots\mathrm{S}_{i_n}\|}\right\}.$$

It is a well-known fact that the joint spectral radius $\hat{\rho}$ plays a critical role in a variety of applications such as switched dynamical systems, differential equations, coding theory, wavelets, combinatorics, and so on; see, for example, [27].

It is easily seen that $\hat{\rho}(\boldsymbol{S})$ is a nonnegative real number, independent of the norm $\|\cdot\|$ used here. Although $\hat{\rho}(\boldsymbol{S})$ is independent of the norm $\|\cdot\|$ chosen, its approximation based on the above limit definition does rely upon an explicit choice of the norm $\|\cdot\|$. How to construct an appropriate norm to realize $\hat{\rho}(\boldsymbol{S})$ has been becoming an important and challenging topic, see e.g. [42]. In many cases, computing $\hat{\rho}$ by definition cannot halt at some finite-time step n, as shown by the single matrix system

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

where $\hat{\rho}(A) = \lim_{n\to+\infty}\sqrt[n]{\|A^n\|} = 1$ by the classical Gel'fand spectral radius formula, however, there holds $\sqrt[n]{\|A^n\|} > 1$ for all $n \geq 1$. For that reason in part, Daubechies and Lagarias in 1992 [16] defined the equally important concept – *generalized spectral radius of* $\boldsymbol{S}$ – by

$$\rho(\boldsymbol{S}) = \limsup_{n\to+\infty}\left\{\max_{(i_1,\ldots,i_n)\in\mathbb{K}^n}\sqrt[n]{\rho(\mathrm{S}_{i_1}\cdots\mathrm{S}_{i_n})}\right\},$$

where $\rho(A)$ stands for the usual spectral radius for any matrix $A \in \mathbb{R}^{d\times d}$. And they conjectured there that a Gel'fand-type formula should hold for $\boldsymbol{S}$. This was proved by Berger and Wang in 1992 [2], i.e., there holds the following Gel'fand-type formula.

Berger–Wang Formula 1.1. $\rho(\boldsymbol{S}) = \hat{\rho}(\boldsymbol{S})$, *for any bounded subset* $\boldsymbol{S} \subset \mathbb{R}^{d\times d}$.

Because of its importance, this Gel'fand-type spectral-radius formula has been reproved by using different interesting approaches, for example, in [17,39,7,5,9,11], in order to gain the inside characteristics of this relationship. According to this formula, the computation of $\rho(\boldsymbol{S})$ has more flexibility,