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Stability and D-stability of the family of real polynomials

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ABSTRACT

The study considers a family of real polynomials with coefficients polynomially dependent on real parameters and presents the necessary and sufficient conditions for the whole family of real polynomials to have no real roots. These results are used to study families of polynomials on stability and D-stability. Several numerical examples are presented.

1. Introduction

Consider a polynomial $f(z)$ with real coefficients ($f(z) \in \mathbb{R}[z], z = x + iy, x, y \in \mathbb{R}$). For given real polynomial $g(x, y)$ ($g(x, y) \in \mathbb{R}[x, y]$), we call $f(z)$ to be a D-stable polynomial if all its roots satisfy the inequality $g(x, y) < 0$. (It is assumed that the domain defined by this inequality in the complex plane is nonempty.)[1] The D-stability problem consists in finding conditions for the coefficients of $f(z)$ providing its D-stability property. The stated problem originated in the work by Hermite [12] and was successfully solved by Routh [24], Hurwitz [13], Schur [25], Cohn [6], and others [24,13,25,6], see also the review [19], for different choices of $g(x, y)$ with a special focus on the cases of $g(x, y) \equiv -y$, $g(x, y) \equiv x$ and $g(x, y) \equiv x^2 + y^2 - 1$, which are important for establishing the asymptotic behavior of the solutions of ordinary differential and, respectively, difference equations. Stabilities with respect to a half-plane and the unit circle are important also for the stability and bifurcation theory, as well as for the control theory.

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[1] Note that the D-stability considered in the paper is different from the matrix D-stability.

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One of the most significant results is the one found by Kharitonov [16] regarding the stability of interval polynomials, i.e., the polynomials of the form

$$f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n, \qquad (a_0 \neq 0)$$

with the coefficients, lying in certain prescribed intervals

$$\underline{a}_i \leq a_i \leq \overline{a}_i, \qquad i = 0, 1, \ldots, n.$$

Kharitonov showed that it is necessary and sufficient to test only four special members of the polynomial family to find out if all polynomials of the family have their roots in the left half-plane of the complex plane (i.e., that they are stable). In the paper [17], this result was extended to the polynomials with complex coefficients. In this case, eight test polynomials are required.

Similar problems regarding entire functions also occur in applications. For example, in control engineering in time delay models, quasi-polynomials of the form

$$F(s) = f_0(s) + e^{sT_1}f_1(s) + \cdots + e^{sT_m}f_m(s)$$

are investigated. The stability of the feedback system with multiple delays

$$\frac{dY}{dt} = AY(t) + \sum_{r=1}^{q} BY(t - \tau_r)$$

is equivalent to the location of all the roots of the entire function $F(s)$ in the left half-plane of the complex plane. More results in this area are presented in [8,18,22].

In the present paper, we deal with a more general problem of the stability of the polynomial with coefficients polynomially dependent on real parameters. We derive necessary and sufficient conditions of stability and D-stability for such a family. (Another approach to the problem of stability based on concepts from sign-definite decomposition can be found in [15].) The method gives the correct result for the one that fails the method described in [1], as can be seen from the example.

Consider now the real polynomial $f(z, \mu_1, \mu_2, \ldots, \mu_k)$ with coefficients polynomially dependent on parameters $\mu_1, \ldots, \mu_k$. We treat the domain of variations of these parameters to be a real box

$$\mathfrak{B} = \{(\mu_1, \mu_2, \ldots, \mu_k) | \underline{\mu}_1 \leq \mu_1 \leq \overline{\mu}_1, \underline{\mu}_2 \leq \mu_2 \leq \overline{\mu}_2, \ldots, \underline{\mu}_k \leq \mu_k \leq \overline{\mu}_k\}.$$

We suppose that the leading coefficient of the polynomial $f(z, \mu_1, \ldots, \mu_k)$ does not vanish at any point in the box $\mathfrak{B}$.

D-**stability problem.** Find the conditions for every polynomial from the family

$$p = \{f(z, \mu_1, \mu_2, \ldots, \mu_k) | (\mu_1, \ldots, \mu_k) \in \mathfrak{B}\} \tag{1}$$

to be D-stable.

This problem was investigated in [7,10], and some approaches for its solution were proposed there. We solve it for a general case, and our approach is based on the theory of differentiable maps [2] and on the Elimination Theory for algebraic equation systems [3,5,27].

Some results from the theory of differentiable maps [2,26] will be sketched in the next section.

2. Algebraic preliminaries

Consider a smooth map $\varphi : M \to N$, where M and N are differentiable manifolds with $\dim M = m$ and $\dim N = n$. Let T_xM be a tangent space of the manifold M at the point x, and let $T_{\varphi(x)}N$ be a tangent space of the manifold N at the point $\varphi(x)$. Then the derivative of the map φ at the point x is a linear map of T_xM into $T_{\varphi(x)}N$:

$$\varphi_{*x} : T_xM \to T_{\varphi(x)}N. \tag{2}$$