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Reversible complex hyperbolic isometries ☆

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ABSTRACT

Let $\mathrm{PU}(n,1)$ denote the isometry group of the n-dimensional complex hyperbolic space $\mathbf{H}_{\mathbb{C}}^n$. An isometry g is called *reversible* if g is conjugate to g^{-1} in $\mathrm{PU}(n,1)$. If g can be expressed as a product of two involutions, it is called *strongly reversible*. We classify reversible and strongly reversible elements in $\mathrm{PU}(n,1)$. We also investigate reversibility and strong reversibility in $\mathrm{SU}(n,1)$.

1. Introduction

An element g in a group G is called *reversible* if there exists $h \in G$ such that $g^{-1} = hgh^{-1}$. The terminology 'real' has also been used in the literature to refer to the reversible elements, for example, see [9,25,11]. If h is an involution, that is $h^{-1} = h$, then this equation becomes $g^{-1} = hgh$ or equivalently $(hg)^2 = hghg = e$, the identity element. In other words, g can be decomposed as the product of two involutions h and hg. In this case g is called *strongly reversible*.

Reversible group elements have been studied in several contexts, for example see [9,19,20,25–27]. The strongly reversible elements are also studied in several contexts, for example see [3–5,7,6,15–17,21,29]. Some of these authors have used the terminology 'strongly real' or 'bireflectional' to refer to strongly reversible elements. From a representation theoretic point of view, the terminology 'real'

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is motivated by a theorem of Frobenius and Schur (1906) which says that if G is finite, the number of real-valued complex irreducible characters of G equals the number of real conjugacy classes of G, cf. [14]. On the other hand from geometric point of view, the terminology 'reversible' is more commonly used, cf. [18,22–24]. We will use the terminology 'reversible' and 'strongly reversible'.

Reversible elements in real hyperbolic geometry have been investigated in many contexts. Let $I\left(\mathbf{H}^n_{\mathbb{R}}\right)$ denote the full isometry group of the n-dimensional real hyperbolic space and let $I_o\left(\mathbf{H}^n_{\mathbb{R}}\right)$ denote the identity component, which is the group of orientation preserving isometries of $\mathbf{H}^n_{\mathbb{R}}$. When $n = 2$ it is well known that every element of $I\left(\mathbf{H}^2_{\mathbb{R}}\right)$ is strongly reversible (and so also reversible) but that there are elements of $I_o\left(\mathbf{H}^2_{\mathbb{R}}\right) = \mathrm{PSL}(2, \mathbb{R})$ that are not reversible. For example $z \longmapsto z + 1$ is not conjugate in $\mathrm{PSL}(2, \mathbb{R})$ to its inverse, $z \longmapsto z - 1$. Things are slightly different for $n = 3$. On page 47 of [10] Fenchel shows that every element of the group $I_o\left(\mathbf{H}^3_{\mathbb{R}}\right) = \mathrm{PSL}(2, \mathbb{C})$ is strongly reversible. On page 51 of [10] he also shows that every element of $I\left(\mathbf{H}^3_{\mathbb{R}}\right)$ is strongly reversible. In higher dimensions, it follows from [13, Theorem 1.2] that every element of $I\left(\mathbf{H}^n_{\mathbb{R}}\right)$ is strongly reversible, also see [3,15,16,21,29]. The reversible elements in $I_o\left(\mathbf{H}^n_{\mathbb{R}}\right)$ have been classified in [12,24], also see [18]. In [12], the first author obtained a linear-algebraic classification by identifying the orientation-preserving isometry group with $\mathrm{SO}_o(n, 1)$. In [24], a geometric classification of the reversible elements in $I_o\left(\mathbf{H}^n_{\mathbb{R}}\right)$ was obtained using the ball model of the hyperbolic space.

Let $\mathbf{H}^n_{\mathbb{C}}$ denote the n-dimensional complex hyperbolic space. Let $I\left(\mathbf{H}^n_{\mathbb{C}}\right)$ denote the full isometry group which consists of holomorphic, as well as anti-holomorphic isometries. The group of all holomorphic isometries can be identified with the projective unitary group $\mathrm{PU}(n, 1)$ which is an index 2 subgroup of $I\left(\mathbf{H}^n_{\mathbb{C}}\right)$. Falbel and Zocca [8] proved that every element in $\mathrm{PU}(2, 1)$ can be expressed as a product of two anti-holomorphic involutions, and so is strongly reversible in $I(\mathbf{H}^2_{\mathbb{C}})$. Choi [2] extended this result to the isometries of $\mathbf{H}^n_{\mathbb{C}}$. It follows from these results that every holomorphic isometry of $\mathbf{H}^n_{\mathbb{C}}$ is reversible in $I\left(\mathbf{H}^n_{\mathbb{C}}\right)$.

In this paper we restrict ourselves to the group $\mathrm{PU}(n, 1)$ and ask for reversible and strongly reversible elements in $\mathrm{PU}(n, 1)$. However, for convenience, we work with the linear group $\mathrm{U}(n, 1)$. We also investigate reversibility and strong reversibility in $\mathrm{SU}(n, 1)$. Earlier, strongly reversible and reversible elements in unitary groups over a field $\mathbb{F}$ have been investigated by Djokovich [3] and Singh–Thakur [25] respectively. It is desirable to have an explicit and actual classification, not just characterisation, of the reversible elements in unitary groups over the complex numbers. Such a classification is not known in general. However, for the groups $\mathrm{U}(n, 1)$ and $\mathrm{SU}(n, 1)$ which are of interest to complex hyperbolic geometry, we have a very satisfactory answer to the classification problem of reversible elements. In this paper we offer a complete classification of the reversible and strongly reversible elements in $\mathrm{U}(n, 1)$, in $\mathrm{SU}(n, 1)$ or in $\mathrm{PU}(n, 1)$. Most of our results are linear algebraic in nature. So people who are not familiar with complex hyperbolic geometry should think of our results as being about unitary groups with respect to an indefinite Hermitian form. The main results of the paper are Theorems 4.1, 4.2 and 4.5 in Section 4. As a consequence we have the following.

Theorem 1.1. *Let T be an element in* $\mathrm{SU}(n, 1)$.

(i) Let T be hyperbolic. Then T is reversible in $\mathrm{SU}(n, 1)$ *if and only if the characteristic polynomial of T has real coefficients.*

(ii) Let T be elliptic. Then T is reversible in $\mathrm{SU}(n, 1)$ *if and only if the characteristic polynomial of T has real coefficients and the eigenvalue of negative or indefinite type of T is* 1 *or* -1.

(iii) Let $T = NA$ be parabolic. Then T is reversible in $\mathrm{SU}(n, 1)$ *if and only if the characteristic polynomial of T has real coefficients and the null eigenvalue of T is* 1 *or* -1 *and the minimal polynomial of N is* $(x - 1)^3$.