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Some inequalities for unitarily invariant norms

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ABSTRACT

We shall prove the inequalities

$$|||(A+B)(A+B)^*||| \leqslant |||AA^* + BB^* + 2AB^*||| \leqslant |||(A-B)(A-B)^* + 4AB^*|||$$

for all $n \times n$ complex matrices A, B and all unitarily invariant norms $|||\cdot|||$. If further A, B are positive definite it is proved that

$$\prod_{j=1}^{k} \lambda_j(A\sharp_\alpha B) \leqslant \prod_{j=1}^{k} \lambda_j(A^{1-\alpha}B^\alpha), \quad 1 \leqslant k \leqslant n,\ 0 \leqslant \alpha \leqslant 1,$$

where $\sharp_\alpha$ denotes the operator means considered by Kubo and Ando and $\lambda_j(X)$, $1 \leqslant j \leqslant n$, denote the eigenvalues of X arranged in the decreasing order whenever these all are real. A number of inequalities are obtained as applications.

1. Introduction

Let $n \in \mathbb{N}$. We shall denote by $\mathcal{M}_n$ the set of $n \times n$ complex matrices. The set of all Hermitian positive semidefinite matrices in $\mathcal{M}_n$ shall be denoted by $\mathcal{S}_n$ whereas $\mathcal{P}_n$ shall denote the set of Hermitian positive definite matrices in $\mathcal{M}_n$. We denote by I_n the identity matrix in $\mathcal{M}_n$. By $X \geqslant Y$ $(X > Y)$ we mean that $X - Y$ is Hermitian positive semidefinite (Hermitian positive definite).

For $X \in \mathcal{M}_n$, we shall always denote by $\lambda_1(X) \geqslant \lambda_2(X) \geqslant \cdots \geqslant \lambda_n(X)$, the eigenvalues of X arranged in the decreasing order whenever these all are real. For $P \in \mathcal{S}_n$, $P^{1/2}$ is the unique Hermitian positive semidefinite square root of P. P^α, $0 \leqslant \alpha \leqslant 1$, is defined similarly (see [6]). By $s_1(X) \geqslant s_2(X) \geqslant \cdots \geqslant s_n(X)$, we denote the eigenvalues of $|X| = (X^*X)^{1/2}$, i.e, singular values of X. Notation $\Re e\, X$ is used for the matrix $(X + X^*)/2$ and is called real part of X.

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Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be elements in $\mathbb{R}^n$. Let $x^{\downarrow}$ and $x^{\uparrow}$ be the vectors obtained by rearranging the coordinates of x in decreasing and increasing order respectively. The weak majorization relation $x \prec_w y$ means

$$\sum_{j=1}^{k} x_j^{\downarrow} \leqslant \sum_{j=1}^{k} y_j^{\downarrow}, \quad 1 \leqslant k \leqslant n,$$

whereas weak log-majorization relation $x \prec_{wlog} y$ means

$$\prod_{j=1}^{k} x_j^{\downarrow} \leqslant \prod_{j=1}^{k} y_j^{\downarrow}, \quad 1 \leqslant k \leqslant n.$$

If $x, y \in \mathbb{R}_+^n$ then it is well known that $x \prec_{wlog} y$ implies $x \prec_w y$.

A norm $|||\cdot|||$ on $\mathcal{M}_n$ is said to be unitarily invariant if $|||UXV||| = |||X|||$ for $X \in \mathcal{M}_n$ and all unitaries $U, V \in \mathcal{M}_n$. The Ky Fan norms given by

$$||X||_{(k)} = \sum_{j=1}^{k} s_j(X), \quad 1 \leqslant k \leqslant n,$$

and p-norms,

$$||X||_p = \left(\sum_{j=1}^{n} (s_j(X))^p \right)^{1/p}, \quad p \geqslant 1,\ X \in \mathcal{M}_n,$$

are familiar examples of unitarily invariant norms. The operator norm $||\cdot||$ is given by $||X|| = s_1(X)$. It is customary to assume a normalization condition that $|||\mathrm{diag}(1, 0, \ldots, 0)||| = 1$. Fan dominance theorem states that $||A||_{(k)} \leqslant ||B||_{(k)}$, $1 \leqslant k \leqslant n$, if and only if $|||A||| \leqslant |||B|||$ for all unitarily invariant norms $|||\cdot|||$. The reader is referred to [2] for more properties of such norms.

If z and w are complex numbers, then we have the following inequality:

$$(z+w)\overline{(z+w)} \leqslant |z\bar{z} + w\bar{w} + 2z\bar{w}| \leqslant \left|(z-w)\overline{(z-w)} + 4z\bar{w}\right|. \tag{1.1}$$

On taking $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ one can see that the inequalities

$$(A+B)(A+B)^* \leqslant \left|AA^* + BB^* + 2AB^*\right| \leqslant \left|(A-B)(A-B)^* + 4AB^*\right|$$

are not true. However in Section 2 we shall prove that

$$|||(A+B)(A+B)^*||| \leqslant |||AA^* + BB^* + 2AB^*||| \leqslant |||(A-B)(A-B)^* + 4AB^*|||$$

for all $A, B \in \mathcal{M}_n$ and all unitarily invariant norms $|||\cdot|||$. In fact we shall prove more general results.

Kubo and Ando [8] considered the geometric mean $\sharp_\alpha$ of two matrices $A, B \in \mathcal{P}_n$, $0 \leqslant \alpha \leqslant 1$, defined by

$$A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}.$$

It is well known that $A\sharp_\alpha B \leqslant \alpha A + (1-\alpha)B$. In [7] Kosem proved the inequality

$$\left|\left|\left|(A\sharp_{1/2}B)^2\right|\right|\right| \leqslant \left|\left|\left|B^{1/2}AB^{1/2}\right|\right|\right|,$$

for $A, B \in \mathcal{P}_n$. We shall prove that for $A, B \in \mathcal{P}_n$ and $0 \leqslant \alpha \leqslant 1$,

$$\prod_{j=1}^{k} \lambda_j(A\sharp_\alpha B) \leqslant \prod_{j=1}^{k} \lambda_j(A^{1-\alpha}B^{\alpha}), \quad 1 \leqslant k \leqslant n. \tag{1.2}$$