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A satellite of the grand Furuta inequality and its application

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ABSTRACT

The grand Furuta inequality has the following satellite (SGF;$t \in [0,1]$), given as a mean theoretic expression:

$$A \geqslant B > 0, t \in [0,1] \Rightarrow A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leqslant B \quad \text{for } r \geqslant t;\ \ p, s \geqslant 1,$$

where $\#_\alpha$ is the α-geometric mean and $\natural_s$ ($s \notin [0,1]$) is a formal extension of $\#_\alpha$. It is shown that (SGF; $t \in [0,1]$) has the Löwner–Heinz property, i.e. (SGF; $t = 1$) implies (SGF;t) for every $t \in [0,1]$. Furthermore, we show that a recent further extension of (GFI) by Furuta himself has also the Löwner–Heinz property.

1. Introduction

An operator means a bounded linear operator acting on a Hilbert space. The usual order $A \geqslant B$ among selfadjoint operators on H is defined by $(Ax, x) \geqslant (Bx, x)$ for $x \in H$. In particular, A is said to be positive and denoted by $A \geqslant 0$ if $(Ax, x) \geqslant 0$ for $x \in H$.

The noncommutativity of operators reflects on the order preservation, the fundamental fact being the Löwner–Heinz inequality:

$$A \geqslant B \geqslant 0 \Rightarrow A^p \geqslant B^p \tag{LH}$$

if and only if $p \in [0,1]$, e.g. [22,15]. Based on this, the theory of operator inequalities has been developed. The points in it might be operator means, the Furuta inequality and the Ando-Hiai inequality, see

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[21,12,1], respectively. A supreme example of their fruits is the grand Furuta inequality (GFI) obtained in [14], see also [5,6,9,17,24]. Specifically it was presented as an interpolation between the Furuta inequality and the Ando-Hiai one. For the reader's convenience, we recall the weighted geometric mean $\#_\alpha$ for $\alpha \in [0, 1]$ and a related binary operation $\natural_s$ for $s \notin [0, 1]$:

$$A \#_\alpha B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}} \quad \text{and} \quad A \natural_s B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^s A^{\frac{1}{2}}.$$

Grand Furuta inequality (GFI).

$$A \geqslant B > 0, \quad t \in [0, 1] \Rightarrow A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leqslant A \quad \text{for } r \geqslant t;\ p, s \geqslant 1.$$

We should mention that the case $t = 0$ in (GFI) is the Furuta inequality, see also [2,3,13,10,23]:

Furuta inequality (FI).

$$A \geqslant B > 0, \quad t \in [0, 1] \Rightarrow A^{-r} \#_{\frac{1+r}{p+r}} B^p \leqslant A \quad \text{for } r \geqslant 0 \text{ and } p \geqslant 1.$$

Closely related to (FI), we note a satellite of it due to Kamei [20]:

$$A \geqslant B > 0, \quad t \in [0, 1] \Rightarrow A^{-r} \#_{\frac{1+r}{p+r}} B^p \leqslant B \, (\leqslant A) \quad \text{for } r \geqslant 0 \text{ and } p \geqslant 1. \tag{SF}$$

The mean theoretic expression of (GFI) above induces the following improvement of it by virtue of (SF), see [6, Theorem 2] and also [11, Theorem]:

Satellite of Grand Furuta inequality (SGF).

$$A \geqslant B > 0, \quad t \in [0, 1] \Rightarrow A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leqslant B \quad \text{for } r \geqslant t;\ p, s \geqslant 1.$$

Now we turn to the Ando-Hiai inequality;

$$X \#_\alpha Y \leqslant 1 \Rightarrow X^r \#_\alpha Y^r \leqslant 1 \quad \text{for } r \geqslant 1, \tag{AH}$$

which is one of the origins of (GFI) and just the case where $r = s$ and $t = 1$ in (GFI). Recently we proposed the 2-variables version of (AH) in [7], see also [3,4]:

Generalized Ando-Hiai inequality (GAH).

$$X \#_\alpha Y \leqslant 1 \Rightarrow X^r \#_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} Y^s \leqslant 1 \quad \text{for } r, s \geq 1.$$

Related to this, we pointed out in [8] that (GAH) is equivalent to (GFI) for $t = 1$, and that so is (FI) to (GFI) for $s = t = 1$. Furthermore a satellite version of (GAH) is given by

$$X \#_\alpha Y \leqslant 1 \Rightarrow X^r \#_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} Y^s \leqslant X \#_\alpha Y \quad \text{for } r, s \geqslant 1. \tag{SGAH}$$

As a matter of fact, this has been shown in the proof of (GAH) itself, by using (SF) twice. As well as (GAH) $\Leftrightarrow$ (GFI; $t = 1$), we can check that (SGAH) is equivalent to (SGF; $t = 1$), in which the correspondence between them is given by $A = X, B = (X^{\frac{1}{2}} Y X^{\frac{1}{2}})^\alpha$ and $\alpha = \frac{1}{p}$. Consequently, we note that (SGF; $t = 1$) has been proved under the assumption of (SGAH).

Under such situation, we would like to point out that the case $t = 1$ is essential in the proof of (GFI). As a matter of fact, we prove that (SGF; $t \in [0, 1]$) has the Löwner–Heinz property, i.e. (SGF; $t = 1$) implies (SGF; t) for every $t \in [0, 1]$. Furthermore, we show that a recent further extension of (GFI) by Furuta himself has also the Löwner–Heinz property.