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Homogeneous tri-additive forms and derivations

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ABSTRACT

Gleason [A.M. Gleason, The definition of a quadratic form, Amer. Math. Monthly 73 (1966) 1049–1066] determined all functionals Q on K-vector spaces satisfying the parallelogram law $Q(\mathbf{x}+\mathbf{y})+Q(\mathbf{x}-\mathbf{y})=2Q(\mathbf{x})+2Q(\mathbf{y})$ and the homogeneity $Q(\lambda\mathbf{x})=\lambda^2Q(\mathbf{x})$. Associated with Q is a unique *symmetric* bi-additive form S such that $Q(\mathbf{x})=S(\mathbf{x},\mathbf{x})$ and $4S(\mathbf{x},\mathbf{y})=Q(\mathbf{x}+\mathbf{y})-Q(\mathbf{x}-\mathbf{y})$. Homogeneity of Q corresponds to that of S: $S(\lambda\mathbf{x},\lambda\mathbf{y})=\lambda^2S(\mathbf{x},\mathbf{y})$. The associated S is not necessarily bi-linear.

Let V be a vector space over a field K, $\mathrm{char}(K)\neq 2,3$. A *tri-additive form* T on V is a map of V^3 into K that is additive in each of its three variables. T is *homogeneous of degree* 3 if $T(\lambda\mathbf{x},\lambda\mathbf{y},\lambda\mathbf{z})=\lambda^3T(\mathbf{x},\mathbf{y},\mathbf{z})$ for all $\lambda\in K,\ \mathbf{x},\mathbf{y},\mathbf{z}\in V$.

We determine the structure of tri-additive forms that are homogeneous of degree 3. One of the keys to this investigation is to find the general solution of the functional equation

$$F(t)+t^3G(1/t)=0,$$

where $F:K\to K$ is additive and $G:K\to K$ is quadratic. It is shown that T is not necessarily tri-linear, even if it is supposed in addition that T is symmetric.

1. Introduction

Let K be a (commutative) field, and let V be a vector space over K. A *bi-additive form* on V is a map of $V\times V$ into K that is additive in each of its two vector variables. Let $n\in\mathbb{N}$. A map $f:V\to K$ is *homogeneous of degree n* if $f(\lambda\mathbf{x})=\lambda^nf(\mathbf{x})$ for all $\lambda\in K,\ \mathbf{x}\in V$.

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Gleason [2] determined all functionals Q on K-vector spaces such that Q obeys the parallelogram law $Q(\mathbf{x}+\mathbf{y})+Q(\mathbf{x}-\mathbf{y})=2Q(\mathbf{x})+2Q(\mathbf{y})$ and is homogeneous of degree 2. There is a one-to-one correspondence between functionals Q satisfying the parallelogram law and *symmetric* bi-additive forms S that is provided by $Q(\mathbf{x})=S(\mathbf{x},\mathbf{x})$ and $4S(\mathbf{x},\mathbf{y})=Q(\mathbf{x}+\mathbf{y})-Q(\mathbf{x}-\mathbf{y})$. Thus the result of Gleason could be framed in terms of symmetric bi-additive forms that are homogeneous of degree 2. There are many other related results, descriptions of which can be found in [4,5,1] and their references.

Our purpose in this paper is to seek similar results when the degrees of additivity and homogeneity are raised to 3. A *tri-additive form* on V is a map of $V\times V\times V$ into K that is additive in each of its three variables. We investigate the structure of tri-additive forms that are homogeneous of degree 3. We do not suppose that our tri-additive forms are symmetric.

One of the keys to this investigation is to find the general solution of the functional equation

$$F(t)+t^3G(1/t)=0, \tag{1}$$

where $F:K\to K$ is additive and $G:K\to K$ is quadratic. We assume throughout that K is of characteristic different from 2, and eventually we will exclude characteristic 3 as well. We find that derivations and second order derivations play an important role in the structure of homogeneous tri-additive forms.

Let R be a commutative ring. A *(first order) derivation* is an additive map A from R into itself which satisfies also $A(xy)=xA(y)+yA(x)$, or equivalently $A(x^2)=2xA(x)$. A map $D:R\to R$ is called a *second order derivation* if D is additive and satisfies

$$D(xyz)=xD(yz)+yD(xz)+zD(xy)-[xyD(z)+xzD(y)+yzD(x)]$$

for all $x,y,z\in R$. Clearly, $D(1)=0$ follows if R has a unity 1.

2. General solution of Eq. (1)

Lemma 2.1. *If additive F and quadratic G satisfy (1), then there exists an additive map $A:K\to K$ such that*

$$F(t)=2A(t)-3tA(1), \tag{2}$$

$$G(t)=3t^2A(1)-2t^3A(t^{-1}), \tag{3}$$

and

$$A(t^2)-3tA(t)+3t^2A(1)-t^3A(t^{-1})=0. \tag{4}$$

Moreover the unique symmetric bi-additive form S associated with G is given by

$$S(t,u)=3tA(u)+3uA(t)-2A(tu)-3tuA(1). \tag{5}$$

Proof. We begin with the simple algebraic identity

$$(1-t)^{-1}(1-u)^{-1}-(1-t)^{-1}=(1-t)^{-1}tu(1-u)^{-1}+u(1-u)^{-1},$$

valid for all $t,u\neq 1$ in K. Applying F to this identity we get

$$F((1-t)^{-1}(1-u)^{-1})-F((1-t)^{-1})=F((1-t)^{-1}tu(1-u)^{-1})+F(u(1-u)^{-1}),$$

which by (1) yields

$$\begin{aligned}&(1-t)^{-3}(1-u)^{-3}G((1-t)(1-u))-(1-t)^{-3}G(1-t)\\ &\quad=(1-t)^{-3}t^3u^3(1-u)^{-3}G((1-t)t^{-1}u^{-1}(1-u))+u^3(1-u)^{-3}G(u^{-1}(1-u))\end{aligned}$$

for all $t,u\neq 0,1$. Multiplying by $(1-t)^3(1-u)^3$ we get