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Extension of Rotfel'd Theorem ☆

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ABSTRACT

Let $f(t)$ be a non-negative concave function on the positive half-line. Given an arbitrary partitioned positive semi-definite matrix, we show that

$$\left\| f\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right) \right\| \le \|f(A)\| + \|f(B)\|$$

for all symmetric (i.e. unitarily invariant) norms. This characterization of concave functions extends a famous trace inequality of Rotfel'd,

$$\operatorname{Tr} f(A+B) \le \operatorname{Tr} f(A) + \operatorname{Tr} f(B)$$

and contains several classical matrix inequalities.

1. Rotfel'd Theorem and norms

In the late 1960s, Rotfel'd [9] proved a famous subadditivity result for concave functions of sums of singular values for operators. This short, self-contained note points out a simple proof of a norm version of this theorem and next, in the second section, extends it to an interesting new block-matrix inequality.

The simplest and most important case of the Rotfel'd Theorem may be stated as a trace inequality:

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Let $A,\ B \geq 0$ *and let* $f(t)$ *be a non-negative concave function on* $[0,\infty)$. *Then,*

$$\mathrm{Tr}\, f(A+B) \leq \mathrm{Tr}\, f(A) + \mathrm{Tr}\, f(B). \tag{1.1}$$

For $f(t) = t^p$ $(0 < p < 1)$ this was first noted by McCarthy [8]. Here and throughout this note, upper case letters stand for n-by-n matrices, or equivalently operators on a finite dimensional Hilbert space $\mathcal{H}$. Inequality (1.1) may be extended to some natural and widely used norms on operators. By a natural norm $\|\cdot\|$, we require the contractive property

$$\|XAY\| \leq \|A\|$$

for all A and all contractions $X,\ Y$ (a contraction maps the unit ball of $\mathcal{H}$ into itself). As extreme points of contractions are unitary, this condition is equivalent to the unitary invariance property

$$\|UAV\| = \|A\|$$

for all A and all unitary operators U, V. It then follows that such a norm is a symmetric gauge function of the singular values (von Neumann's theorem). Therefore we call it a symmetric norm; though the term of unitarily invariant norm is also common. Typical examples are the Schatten p-norms (the l_p norms of the singular values), especially the trace norm and the operator norm, and the Ky Fan k-norms, $A \mapsto \|A\|_{(k)}$, defined as the sum of the k largest singular values. Hence from the min–max principle follows:

$\|A\|_{(k)} = \max_P \mathrm{Tr}\, |A|P$ where the maximum run over all rank k projections P.

Here $|A|$ is the positive part in the polar decomposition $A = U|A|$. This simple principle is known as Ky Fan's maximal principle. Another principle of Ky Fan is the dominance one:

$\|A\|_{(k)} \geq \|B\|_{(k)}$ for all k ensures that $\|A\| \geq \|B\|$ for all symmetric norms.

These principles, norms, gauge function and majorization theory are quite well exposed in several books, for example in [2].

It is easy to derive (1.2) from (1.1) in case of the Ky Fan norms, and next for all symmetric norms thanks to the dominance principle; the following proof is much simpler than the one of [11], Theorem 4.4.

Let $A,\ B \geq 0$ *and let* $f(t)$ *be a non-negative concave function on* $[0,\infty)$. *Then, for all symmetric norms,*

$$\|\, f(A+B)\, \| \leq \|f(A)\| + \|f(B)\|. \tag{1.2}$$

Proof. We will derive (1.2) from (1.1). A simple proof of (1.1) is given in the next section. It suffices to prove (1.2) for the Ky Fan k-norms,

$$\|\, f(A+B)\, \|_{(k)} \leq \|f(A)\|_{(k)} + \|f(B)\|_{(k)}. \tag{1.3}$$

Indeed, since $\|f(A)\|_{(k)} + \|f(B)\|_{(k)} = \|f(A')+f(B')\|_{(k)}$, where X' is the diagonal matrix whose entries down the diagonal are the eigenvalues of the positive matrix X, the Ky Fan dominance principle and (1.3) ensure that

$$\|\, f(A+B)\, \| \leq \|\, f(A') + f(B')\, \|$$

for all symmetric norms so that (1.2) follows from the triangle inequality and the obvious equalities $\|X\| = \|X'\|$.

Let $g(t) := f(t) - f(0)$. Since for every positive matrix $\|f(X)\|_{(k)} = \|g(X)\|_{(k)} + kf(0)$, it suffices to prove (1.3) when $f(0) = 0$. Letting P denote a spectral projection of $A+B$ corresponding to the k largest eigenvalues, we then have

$$\begin{aligned} \|\, f(A+B)\, \|_{(k)} &= \mathrm{Tr}\, f(P(A+B)P) \\ &\leq \mathrm{Tr}\, f(PAP) + \mathrm{Tr}\, f(PBP) \\ &\leq \|\, f(A)\, \|_{(k)} + \|\, f(B)\, \|_{(k)}, \end{aligned}$$