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Schatten p-norm inequalities related to an extended operator parallelogram law

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ABSTRACT

Let $\mathcal{C}_p$ be the Schatten p-class for $p > 0$. Generalizations of the parallelogram law for the Schatten 2-norms have been given in the following form: if $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$ and $\mathbf{B} = \{B_1, B_2, \ldots, B_n\}$ are two sets of operators in $\mathcal{C}_2$, then

$$\sum_{i,j=1}^{n} \|A_i - A_j\|_2^2 + \sum_{i,j=1}^{n} \|B_i - B_j\|_2^2 = 2\sum_{i,j=1}^{n} \|A_i - B_j\|_2^2 - 2\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_2^2.$$

In this paper, we give generalizations of this as pairs of inequalities for Schatten p-norms, which hold for certain values of p and reduce to the equality above for $p = 2$. Moreover, we present some related inequalities for three sets of operators.

1. Introduction

Suppose that $\mathbb{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ endowed with an inner product $\langle \cdot, \cdot \rangle$. Let $A \in \mathbb{B}(\mathcal{H})$ be a compact operator and

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let $\{s_j(A)\}$ denote the sequence of decreasingly ordered singular values of A, i.e. the eigenvalues of $|A| = (A^*A)^{1/2}$. The Schatten p-norm (p-quasi-norm, respectively) for $1 \le p < \infty$ ($0 < p < 1$, respectively) is defined by

$$\|A\|_p = \left(\sum_{j=1}^{\infty} s_j^p(A)\right)^{1/p}.$$

For $p > 0$, the Schatten p-class, denoted by $\mathcal{C}_p$, is defined to be the two-sided ideal in $\mathbb{B}(\mathcal{H})$ of those compact operators A for which $\|A\|_p$ is finite. Clearly

$$\left\| \, |A|^2 \, \right\|_{p/2} = \|A\|_p^2 \tag{1.1}$$

for $p > 0$. In particular, $\mathcal{C}_1$ and $\mathcal{C}_2$ are the trace class and the Hilbert–Schmidt class, respectively. For $1 \le p < \infty$, $\mathcal{C}_p$ is a Banach space; in particular the triangle inequality holds. For $0 < p < 1$, the quasi-norm $\|\cdot\|_p$ does not satisfy the triangle inequality, but however satisfies the inequality $\|A+B\|_p^p \leqslant \|A\|_p^p + \|B\|_p^p$. For more information on the theory of Schatten p-norms the reader is referred to [8, Chapter 2].

It follows from [7, Corollary 2.7] that for $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathbb{B}(\mathcal{H})$

$$\sum_{i,j=1}^{n} \|A_i - A_j\|^2 + \sum_{i,j=1}^{n} \|B_i - B_j\|^2 = 2\sum_{i,j=1}^{n} \|A_i - B_j\|^2 - 2\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|^2, \tag{1.2}$$

which is indeed a generalization of the classical *parallelogram law*:

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2 \quad (z, w \in \mathbb{C}).$$

There are several extensions of parallelogram law among them we could refer the interested reader to [2,3,6,9,10]. Generalizations of the parallelogram law for the Schatten p-norms have been given in the form of the celebrated Clarkson inequalities (see [4] and references therein). Since $\mathcal{C}_2$ is a Hilbert space under the inner product $\langle A, B\rangle = \mathrm{tr}(B^*A)$, it follows from an equality similar to (1.2) stated for vectors of a Hilbert space (see Corollary 2.7 [7]) that if $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{C}_2$, then

$$\sum_{i,j=1}^{n} \|A_i - A_j\|_2^2 + \sum_{i,j=1}^{n} \|B_i - B_j\|_2^2 = 2\sum_{i,j=1}^{n} \|A_i - B_j\|_2^2 - 2\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_2^2. \tag{1.3}$$

In [7] a joint operator extension of the Bohr and parallelogram inequalities is presented. In particular, it follows from [7, Corollary 2.3] that if $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathbb{B}(\mathcal{H})$, then

$$\sum_{1\le i<j\le n} |A_i - A_j|^2 + \sum_{1\le i<j\le n} |B_i - B_j|^2 = \sum_{i,j=1}^{n} |A_i - B_j|^2 - \left|\sum_{i=1}^{n}(A_i - B_i)\right|^2. \tag{1.4}$$

In this paper, we give a generalization of the equality (1.3) for the Schatten p-norms ($p > 0$). First we present similar consideration for three sets of operators. In addition, we provide pairs of complementary inequalities that reduce to (1.3) for the certain value $p = 2$.

2. Schatten p-norm inequalities

To accomplish our results, we need the following lemma which can be deduced from [1, Lemma 4] and [8, p. 20]: