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On the continuity of the generalized spectral radius in max algebra

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ABSTRACT

Given a bounded set Ψ of $n \times n$ non-negative matrices, let $\rho(\Psi)$ and $\mu(\Psi)$ denote the generalized spectral radius of Ψ and its max version, respectively. We show that

$$\mu(\Psi) = \sup_{t\in(0,\infty)} \left(n^{-1}\rho\left(\Psi^{(t)}\right)\right)^{1/t},$$

where $\Psi^{(t)}$ denotes the Hadamard power of Ψ. We apply this result to give a new short proof of a known fact that $\mu(\Psi)$ is continuous on the Hausdorff metric space (β, H) of all nonempty compact collections of $n \times n$ non-negative matrices.

1. Introduction

The algebraic system max algebra and its isomorphic versions provide an attractive way of describing a class of non-linear problems appearing for instance in manufacturing and transportation scheduling, information technology, discrete event-dynamic systems, combinatorial optimisation, mathematical physics, DNA analysis, …(see e.g. [8,1,2,7,3,21,28]). Max algebra's usefulness arises from a fact that these non-linear problems become linear when described in the max algebra language.

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Following the notation from [2,10,23,24,17], the max algebra consists of the set of non-negative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product ab, where $a, b \geq 0$. Let $A = [a_{ij}]$ be a $n \times n$ non-negative matrix, i.e., $a_{ij} \geq 0$ for all $i, j = 1, \ldots, n$. We may denote a_{ij} also by $[A]_{ij}$. Let $\mathbb{R}^{n\times n}$ be the set of all $n \times n$ real matrices and $\mathbb{R}_+^{n\times n}$ the set of all $n \times n$ non-negative matrices. The operations between matrices and vectors in the max algebra are defined by analogy with the usual linear algebra. For instance, the product of $A, B \in \mathbb{R}_+^{n\times n}$ in the max algebra is denoted by $A \otimes B$, where $[A \otimes B]_{ij} = \max_{k=1,\ldots,n} a_{ik}b_{kj}$. The notation $A_\otimes^2$ means $A \otimes A$, and $A_\otimes^k$ denotes the k-th max power of A. If $x = [x_i] \in \mathbb{R}^n$ is a non-negative vector, then the notation $A \otimes x$ means $[A \otimes x]_i = \max_{j=1,\ldots,n} a_{ij}x_j$. The usual associative and distributive laws hold in this algebra. Note that the standard products are denoted by AB and Ax.

The weighted directed graph $\mathcal{D}(A)$ associated with A has a vertex set $\{1, 2, \ldots, n\}$ and edges (i, j) from a vertex i to a vertex j with weight a_{ij} if and only if $a_{ij} > 0$. A path of length k is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})$. A circuit of length k is a path with $i_{k+1} = i_1$, where $i_1, i_2, \ldots, i_k$ are distinct. Associated with this circuit is the *circuit geometric mean* known as $(a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1})^{1/k}$. The maximum circuit geometric mean in $\mathcal{D}(A)$ is denoted by $\mu(A)$. Note that circuits (i_1, i_1) of length 1 (loops) are included here and that we also consider empty circuits, i.e., circuits that consist of only one vertex and have length 0. For empty circuits, the associated circuit geometric mean is zero.

There are many different descriptions of the maximum circuit geometric mean $\mu(A)$ (see e.g. [13,14,9,20, p. 366,3, p. 130,10,27,25,30,29,16]). It was proved in [14] that given $A \in \mathbb{R}_+^{n\times n}$

$$\mu(A) = \lim_{t\to\infty} \rho(A^{(t)})^{1/t}, \tag{1}$$

where $A^{(t)} = [a_{ij}^t]$ is a Hadamard (or also Schur) power of A and ρ the spectral radius. Alternative and simplified proofs of (1) can be found in [9,20, p. 366,3, p. 130,10,30]. We also have

$$\mu(A) \leq \rho(A) \leq n\mu(A) \tag{2}$$

(see e.g. [9,20, p. 366,22,23,30]).

It is known that $\mu(A)$ is the largest max eigenvalue of A. Moreover, if A is irreducible, then $\mu(A)$ is the unique max eigenvalue and every max eigenvector is positive (see [2, Theorem 2] and [22, Theorem 1]). We also have

$$\mu(A) = \lim_{k\to\infty} \|A_\otimes^k\|^{1/k} \tag{3}$$

for an arbitrary vector norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ (see [10, Lemma 4.1,22,23,30]).

Given an irreducible non-negative matrix A, algorithms for computing $\mu(A)$ and the max eigenvector x were established in [10–12]. On the other hand, infinite-dimensional generalizations of μ can be found in [27,25,29].

Let Σ be a bounded set of $n \times n$ complex matrices. For $m \geq 1$, let

$$\Sigma^m = \{A_1A_2\cdots A_m : A_i \in \Sigma\}.$$

The generalized spectral radius of Σ is defined by

$$\rho(\Sigma) = \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} \rho(A)]^{1/m}. \tag{4}$$

It was shown in [5] that $\rho(\Sigma)$ is equal to the joint spectral radius of Σ, i.e.,

$$\rho(\Sigma) = \lim_{m\to\infty} [\sup_{A\in\Sigma^m} \|A\|]^{1/m}, \tag{5}$$

where $\|\cdot\|$ is any vector norm on $\mathbb{C}^{n\times n}$. This equality is called the Berger-Wang formula or also the generalized spectral radius theorem. Since then many different type of proofs of (5) were obtained