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Is modified PSD equivalent to modified SOR for two-cyclic matrices?

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ABSTRACT

In this paper we study the convergence analysis of the Modified Preconditioned Simultaneous Displacement (MPSD) method when A is a two-cyclic matrix. Convergence conditions and optimum values of the parameters are determined in case the eigenvalues of the associated Jacobi iteration matrix are either all real or all imaginary.

1. Introduction

The Preconditioned Simultaneous Displacement (PSD) iterative method was introduced in [5] and studied for the numerical solution of the linear system

$$Au = b, \tag{1}$$

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where $A \in \mathbb{C}^{n,n}$ is a nonsingular, sparse matrix with nonvanishing diagonal entries and $u, b \in \mathbb{C}^n$ with b given and u to be determined. PSD is a first order extrapolation of SSOR and as such it was shown to be asymptotically twice as fast as SSOR for the natural ordering [5]. When A in (1) is a two-cyclic matrix, PSD attains a maximum rate of convergence equivalent to Extrapolated Gauss–Seidel method (EGS) method [9] if the associated Jacobi iteration matrix possesses only real eigenvalues (real case), whereas if these eigenvalues are all imaginary (imaginary case), its convergence is improved by an order of magnitude and becomes equal the Extrapolated SOR (ESOR) for the optimum values of its parameters [7]. In [10] the convergence analysis of Modified PSD (MPSD) was studied in the real case under the assumption that zero is an eigenvalue of the Jacobi iteration matrix. It was shown that MPSD is equivalent to SOR in the sense that both methods possess the same rate of convergence. In the present work, we introduce an additional parameter in the MPSD method with the hope of increasing its rate of convergence. Our starting point is the derivation of a functional equation which relates the eigenvalues of the MPSD iteration matrix to its associated two-cyclic Jacobi iteration matrix. Then, we find sufficient and necessary conditions for MPSD to converge as well as determine optimum values of its parameters in case the associated Jacobi iteration matrix possesses either real or imaginary eigenvalues. It is shown that the optimum MPSD method is equivalent to the optimum MSOR method thus answering the question posed by the title of the paper. An analogous result was proven in [8] for the MSSOR and SOR methods only for the real case and assuming that zero is an eigenvalue of the associated Jacobi iteration matrix. The paper is organized as follows. In Section 2, we derive the MPSD functional relationship for two-cyclic matrices. In Section 3, sufficient and necessary conditions are found for MPSD to converge under the assumption that the eigenvalues of the associated Jacobi iteration matrix are either all real or all imaginary. As a by-product of our analysis we find convergence conditions for the Modified Extrapolated SOR (MESOR) and for the Modified SSOR (MSSOR) methods. Under the same assumptions we determine optimum values for the parameters of all the aforementioned methods in Section 4. In Section 5, we present our numerical results. Finally, in Section 6, we state our remarks and conclusions.

2. The functional relationship

Let $A \in \mathbb{C}^{n,n}$ be a two-cyclic and consistently ordered matrix of the form

$$A = \begin{bmatrix} D_1 & H \\ K & D_2 \end{bmatrix}, \tag{2}$$

where $H \in \mathbb{C}^{n_1,n_2}$, $K \in \mathbb{C}^{n_2,n_1}$ and $D_1 \in \mathbb{C}^{n_1,n_1}$, $D_2 \in \mathbb{C}^{n_2,n_2}$ are diagonal nonsingular matrices, respectively with $n_1 + n_2 = n$. Let A possess the splitting

$$A = D - L - U, \tag{3}$$

where

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ -K & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -H \\ 0 & 0 \end{pmatrix}.$$

The Jacobi iteration matrix is

$$B = D^{-1}(L + U) = \widetilde{L} + \widetilde{U} \tag{4}$$

with

$$\widetilde{L} = D^{-1}L = \begin{pmatrix} 0 & 0 \\ \overline{L} & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{U} = D^{-1}U = \begin{pmatrix} 0 & \overline{U} \\ 0 & 0 \end{pmatrix}, \tag{5}$$

where $\overline{L} = -D_2^{-1}K$ and $\overline{U} = -D_1^{-1}H$. For the numerical solution of (1), we consider the iterative scheme

$$u^{(m+1)} = u^{(m)} + R^{-1}T(b - Au^{(m)}), \tag{6}$$

where R is the nonsingular matrix