Linear Algebra and its Applications 432 (2010) 2861–2878

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Maximal exponents of polyhedral cones (II)

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A R T I C L E I N F O

Article history:
Received 9 July 2009
Accepted 17 December 2009
Available online 6 February 2010

Submitted by H. Schneider

AMS classification:
15A48
05C50
47A06

Keywords:
Cone-preserving map
K-primitive matrix
Exponents
Polyhedral cone
Exp-maximal cone
Exp-maximal K-primitive matrix
Cone-equivalence
Minimal cone

A B S T R A C T

Let K be a proper (i.e., closed, pointed, full convex) cone in $\mathbb{R}^n$. An $n \times n$ matrix A is said to be K-primitive if there exists a positive integer k such that $A^k(K \setminus \{0\}) \subseteq \text{int}\, K$; the least such k is referred to as the exponent of A and is denoted by $\gamma(A)$. For a polyhedral cone K, the maximum value of $\gamma(A)$, taken over all K-primitive matrices A, is called the exponent of K and is denoted by $\gamma(K)$. It is proved that the maximum value of $\gamma(K)$ as K runs through all n-dimensional minimal cones (i.e., cones having $n+1$ extreme rays) is $n^2 - n + 1$ if n is odd, and is $n^2 - n$ if n is even, the maximum value of the exponent being attained by a minimal cone with a balanced relation for its extreme vectors. The K-primitive matrices A such that $\gamma(A)$ attain the maximum value are identified up to cone-equivalence modulo positive scalar multiplication.

1. Introduction

This is the second of a sequence of papers studying the maximal exponents of K-primitive matrices over polyhedral cones. Here for a polyhedral (proper) cone K in $\mathbb{R}^n$ by a K-*primitive matrix* we mean a real square matrix A for which there exists a positive integer k such that A^k maps every nonzero vector of K into the interior of K; the least such k, denoted by $\gamma(A)$, is referred to as the *exponent* of A. In

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[1] Supported by National Science Council of the Republic of China.

0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.12.026

[12], the first paper in the sequence, it is proved that if K is an n-dimensional polyhedral cone with m extreme rays then its exponent $\gamma(K)$, which is defined to be $\max\{\gamma(A) : A \text{ is } K\text{-primitive}\}$, does not exceed $(n-1)(m-1)+1$, thus answering in the affirmative a conjecture posed by Steve Kirkland. [When $m = n$, the latter bound reduces to Wielandt's classical sharp bound [20] for exponents of (nonnegative) primitive matrices of a given order]. The general question of what the maximum value of $\gamma(K)$ is, when K is taken over all n-dimensional polyhedral cones with m extreme rays, for a given pair of positive integers m, n, remains unresolved. In this paper we take up the question for the minimal cone case, i.e., when $m = n+1$.

The upper bound $(n-1)(m-1)+1$ for $\gamma(K)$ obtained in [12] may suggest that for n-dimensional minimal cones K, $n^2 - n + 1$ is a sharp upper bound for $\gamma(K)$. It turns out that this is true when n is odd, but for even n the sharp upper bound is one less. In [12], in connection with the equality case of the upper bound $(n-1)(m-1)+1$ (or $(n-1)(m-1)$) for $\gamma(A)$, two special digraphs, represented by Figs. 1 and 2, respectively are singled out. They are precisely the two known primitive digraphs on n vertices (for some n) with the length of the shortest circuit equal to $n-1$. They will play an important role in this work.

We now describe the contents of this paper in some detail.

Section 2 contains most of the definitions, together with the relevant known results, which we need for the paper. For the sake of convenience, we collect together properties/results on minimal cones in Section 3. In particular, we show that for minimal cones, the concepts of “linearly isomorphic" and “combinatorially equivalent" are equivalent.

In Section 4 we prove that the maximum value of $\gamma(K)$ as K runs through all n-dimensional minimal cones is $n^2 - n + 1$ if n is odd, and is $n^2 - n$ if n is even. We also determine (up to linear isomorphism) the minimal cones K (and also the corresponding K-primitive matrices A) such that $\gamma(K)$ (and $\gamma(A)$) attains the maximum value. In particular, it is found that every minimal cone K whose exponent attains the maximum value has a balanced relation for its extreme vectors and also if A is a K-primitive matrix such that $\gamma(A) = \gamma(K)$ then necessarily the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is, up to graph isomorphism, given by Figs. 1 or 2.

In Section 5 we consider the question of uniqueness of the minimal cone K and the corresponding K-primitive matrix A such that $\gamma(K)$ and $\gamma(A)$ attain the maximum value. It is proved that for every integer $n \geqslant 3$, there are (up to linear isomorphism) one or two n-dimensional optimal minimal cones, depending on whether n is odd or even. However, for each such minimal cone K, there are uncountably infinitely many pairwise non-cone-equivalent linearly independent optimal K-primitive matrices.

In Section 6, the final section, we give some open questions.

2. Preliminaries

We take for granted standard properties of nonnegative matrices, complex matrices and graphs that can be found in textbooks (see, for instance, [3,4,8,9,11]). A familiarity with elementary properties of finite-dimensional convex sets, convex cones and cone-preserving maps is also assumed (see, for instance,[2,14,17,21]). To fix notation and terminology, we give some definitions.

Let K be a nonempty subset of a finite-dimensional real vector space V. The set K is called a *convex cone* if $\alpha x + \beta y \in K$ for all $x, y \in K$ and $\alpha, \beta \geqslant 0$; K is *pointed* if $K \cap (-K) = \{0\}$; K is *full* if its interior $\operatorname{int} K$ (in the usual topology of V) is nonempty, equivalently, $K - K = V$. If K is closed and satisfies all of the above properties, K is called a *proper cone*.

In this paper, unless specified otherwise, we always use K to denote a proper cone in the n-dimensional Euclidean space $\mathbb{R}^n$.

We denote by $\geqslant^K$ the partial ordering of $\mathbb{R}^n$ induced by K, i.e., $x \geqslant^K y$ if and only if $x - y \in K$.

A subcone F of K is called a *face* of K if $x \geqslant^K y \geqslant^K 0$ and $x \in F$ imply $y \in F$. If $S \subseteq K$, we denote by $\Phi(S)$ the *face of K generated by S*, that is, the intersection of all faces of K including S. If $x \in K$, we write $\Phi(\{x\})$ simply as $\Phi(x)$. It is known that for any vector $x \in K$ and any face F of K, $x \in \operatorname{ri} F$ if and only if $\Phi(x) = F$; also, $\Phi(x) = \{y \in K : x \geqslant^K \alpha y \text{ for some } \alpha > 0\}$. (Here we denote by $\operatorname{ri} F$ the *relative interior of* F.) A vector $x \in K$ is called an *extreme vector* if either x is the zero vector or x is nonzero and