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Nonlinear Lie derivations of triangular algebras ☆

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ABSTRACT

In this paper we prove that every nonlinear Lie derivation of triangular algebras is the sum of an additive derivation and a map into its center sending commutators to zero.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$, and let $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Recall that a left $\mathcal{A}$-module $\mathcal{M}$ is faithful if $a \in \mathcal{A}$ and $a\mathcal{M} = 0$ implies that $a = 0$. The $\mathcal{R}$-algebra

$$\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

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under the usual matrix operations is called a triangular algebra. The most important examples of triangular algebras are upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras. Cheung [4,5] described commuting maps and Lie derivations of these algebras. Benkovič and Eremita [2] studied commuting traces of biadditive maps and Lie isomorphisms of triangular algebras. Benkovič [3] investigated biderivations of triangular algebras. Wong [19] treated Jordan isomorphisms of triangular algebras, while Zhang and Yu [20] studied Jordan derivations.

Let $\mathcal{A}$ be an algebra on a commutative ring $\mathcal{R}$. A map $\delta : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if it is additive and satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. If there exists an element $a \in \mathcal{A}$ such that $\delta(x) = [x, a]$ for all $x \in \mathcal{A}$, where $[x, a] = xa - ax$ is the Lie product or the commutator of the elements $x, a \in \mathcal{A}$, then δ is said to be an inner derivation. Let $\varphi : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity assumption). We say that φ is a nonlinear Lie derivation if $\varphi([x, y]) = [\varphi(x), y] + [x, \varphi(y)]$ for all $x, y \in \mathcal{A}$.

The structure of additive or linear Lie derivations on rings or algebras has been studied by many authors. For example, see [1,11,13–18,21] and their references. Recently, Cheng and Zhang [6] described nonlinear Lie derivations of upper triangular matrix algebras. In this paper we will investigate nonlinear Lie derivations of triangular algebras.

2. Main result

Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and let $Z(\mathcal{U})$ be its centre. It follows from [4, Proposition 3] that

$$Z(\mathcal{U}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : am = mb \text{ for all } m \in \mathcal{M} \right\}. \quad (1)$$

Let us define two natural projections $\pi_{\mathcal{A}} : \mathcal{U} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{U} \to \mathcal{B}$ by

$$\pi_{\mathcal{A}} : \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mapsto a \quad \text{and} \quad \pi_{\mathcal{B}} : \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mapsto b.$$

Then $\pi_{\mathcal{A}}(Z(\mathcal{U})) \subseteq Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{U})) \subseteq Z(\mathcal{B})$, and there exists a unique algebra isomorphism $\tau : \pi_{\mathcal{A}}(Z(\mathcal{U})) \to \pi_{\mathcal{B}}(Z(\mathcal{U}))$ such that $am = m\tau(a)$ for all $m \in \mathcal{M}$.

Let $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ be identities of the algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, and let 1 be the identity of the triangular algebra $\mathcal{U}$. Throughout this paper we shall use following notation:

$$e_1 = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = 1 - e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}$$

and

$$\mathcal{U}_{ij} = e_i \mathcal{U} e_j \quad \text{for } 1 \leqslant i \leqslant j \leqslant 2.$$

It is clear that the triangular algebra $\mathcal{U}$ may be represented as

$$\mathcal{U} = e_1\mathcal{U}e_1 + e_1\mathcal{U}e_2 + e_2\mathcal{U}e_2 = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{22}. \quad (2)$$

Here $\mathcal{U}_{11}$ and $\mathcal{U}_{22}$ are subalgebras of $\mathcal{U}$ isomorphic to $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\mathcal{U}_{12} \subseteq \mathcal{U}$ is a $(\mathcal{U}_{11}, \mathcal{U}_{22})$-bimodule isomorphic to the bimodule $\mathcal{M}$. We also see that $\pi_{\mathcal{A}}(Z(\mathcal{U}))$ and $\pi_{\mathcal{B}}(Z(\mathcal{U}))$ are isomorphic to $e_1 Z(\mathcal{U}) e_1$ and $e_2 Z(\mathcal{U}) e_2$, respectively. Then there is an algebra isomorphism $\sigma : e_1 Z(\mathcal{U}) e_1 \to e_2 Z(\mathcal{U}) e_2$ such that $am = m\sigma(a)$ for all $m \in \mathcal{U}_{12}$.

In this section, we will prove the following theorem.

Theorem 2.1. *Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and let $\varphi : \mathcal{U} \to \mathcal{U}$ be a nonlinear Lie derivation. If $\pi_{\mathcal{A}}(Z(\mathcal{U})) = Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{U})) = Z(\mathcal{B})$, then φ is the sum of an additive derivation and a map into its center $Z(\mathcal{U})$ sending each commutator to zero.*

Next we assume that $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a triangular algebra with $\pi_{\mathcal{A}}(Z(\mathcal{U})) = Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{U})) = Z(\mathcal{B})$, and that $\varphi : \mathcal{U} \to \mathcal{U}$ is a nonlinear Lie derivation. From Eq. (1), we have the following lemma.

Lemma 2.1. *Let $x \in \mathcal{U}$. Then $x \in \mathcal{U}_{12} + Z(\mathcal{U})$ if and only if $[x, m] = 0$ for all $m \in \mathcal{U}_{12}$.*