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Additive maps derivable or Jordan derivable at zero point on nest algebras

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ABSTRACT

Let Alg$\mathcal{N}$ be a nest algebra associated with the nest $\mathcal{N}$ on a (real or complex) Banach space X. Assume that every $N \in \mathcal{N}$ is complemented whenever $N_- = N$. Let δ : Alg$\mathcal{N} \to$ Alg$\mathcal{N}$ be an additive map. It is shown that the following three conditions are equivalent: (1) δ is derivable at zero point, i.e., $\delta(AB) = \delta(A)B + A\delta(B)$ whenever $AB = 0$; (2) δ is Jordan derivable at zero point, i.e., $\delta(AB + BA) = \delta(A)B + A\delta(B) + B\delta(A) + \delta(B)A$ whenever $AB + BA = 0$; (3) δ has the form $\delta(A) = \tau(A) + cA$ for some additive derivation τ and some scalar c. It is also shown that δ is generalized derivable at zero point, i.e., $\delta(AB) = \delta(A)B + A\delta(B) - A\delta(I)B$ whenever $AB = 0$, if and only if δ is an additive generalized derivation. Finer characterizations of above maps are given for the case $\dim X = \infty$.

1. Introduction

Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Recall that a linear (an additive) map δ from $\mathcal{A}$ into $\mathcal{M}$ is derivable at $Z \in \mathcal{A}$ (or generalized derivable at $Z \in \mathcal{A}$) if $\delta(AB) = \delta(A)B + A\delta(B)$ (or $\delta(AB) = \delta(A)B + A\delta(B) - A\delta(I)B$) for any $A, B \in \mathcal{A}$ with $AB = Z$. Recall that δ is Jordan derivable at $Z \in \mathcal{A}$ if $\delta(AB + BA) = \delta(A)B + A\delta(B) + B\delta(A) + \delta(B)A$ for any $A, B \in \mathcal{A}$ with $AB + BA = Z$.

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There have been a number of papers concerning the study of conditions under which (generalized or Jordan) derivations of operator algebras can be completely determined by the action on some sets of operators. We mention some such results related to this paper. It was shown by Chebotar et al. in [1] that every additive map δ on a prime ring $\mathcal{A}$ containing a nontrivial idempotent has the form $\delta(A) = \tau(A) + CA$ if the map is derivable at zero point, where τ is an additive derivation and C is a central element of $\mathcal{A}$. Note that the nest algebras are important operator algebras that are not prime. Using the main theorems in [3,4], Jing et al. in [9] showed that, for the case of nest algebras on Hilbert spaces, the set of linear maps derivable at zero point with $\delta(I) = 0$ coincides with the set of inner derivations. In [11], Li et al. showed that every linear map δ derivable at zero point with $\delta(I) = 0$ on a nest subalgebra of a factor von Neumann algebra is a derivation. Zhu and Xiong showed in [15] that every norm continuous linear map generalized derivable at zero point between finite nest algebras on Hilbert spaces is a generalized inner derivation. In [16], Zhu and Xiong proved that every norm continuous linear map δ generalized derivable at zero point on finite CSL algebra on a complex separable Hilbert space is a generalized derivation (i.e., $\delta(AB) = \delta(A)B + A\delta(B) - A\delta(I)B$ for all A, B). In [10], Jing showed that every linear map on $\mathcal{B}(H)$ Jordan derivable at unit operator I is an inner derivation, where $\mathcal{B}(H)$ denotes the algebra of all bounded linear operators on Hilbert space H. The purpose of this paper is to continue the discussion of this topic and generalize above results on nest algebras by a different approach. We'll characterize the additive maps that are (generalized) derivable at zero point or Jordan derivable at zero point between nest algebras on Banach spaces, without any continuity assumption on the maps.

Let $\mathcal{N}$ be a nest on a Banach space X over the real or complex field $\mathbb{F}$ with each $N \in \mathcal{N}$ complemented in X whenever $N_- = N$. It is obvious that the nests on Hilbert spaces, finite nests and the nests having order-type $\omega + 1$ or $1 + \omega^*$, where ω is the order-type of the natural numbers, satisfy this condition automatically. Let $\delta : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ be an additive map. In Section 2 we show that, if δ is derivable at zero point, then $\delta(I) = cI$ for some scalar $c \in \mathbb{F}$ and there is an additive derivation τ such that $\delta(A) = \tau(A) + cA$ for all $A \in \mathrm{Alg}\mathcal{N}$; if δ is generalized derivable at zero point, then δ is in fact a generalized derivation. In particular, if $\delta(I) = 0$, then in both cases, δ is an additive derivation (Theorems 2.1 and 2.2). Section 3 is devoted to characterizing additive maps that Jordan derivable at zero point. We show that, if δ is Jordan derivable at zero point, then there exist an additive derivation τ and a scalar c such that $\delta(A) = \tau(A) + cA$ (Theorem 3.1). Thus δ is derivable at zero point if and only if it is Jordan derivable at zero point. Particularly, if X is infinite dimensional, then the following three conditions are equivalent: (1) δ is additive and is derivable at zero point; (2) δ is additive and is Jordan derivable at zero point; (3) δ is linear and there exist an operator $T \in \mathrm{Alg}\mathcal{N}$ and a scalar $c \in \mathbb{F}$ such that $\delta(A) = AT - TA + cA$ for all $A \in \mathrm{Alg}\mathcal{N}$ (Corollary 3.3). We also show that if $\dim X = \infty$, then δ is additive and generalized derivable at zero point if and only if δ is linear and there exist $T, S \in \mathrm{Alg}\mathcal{N}$ such that $\delta(A) = AT + SA$ for all $A \in \mathrm{Alg}\mathcal{N}$ (Theorem 2.4). For the case $\dim X < \infty$, if δ is continuous on $\mathbb{F}I$, then δ is additive and (Jordan) derivable (resp. generalized derivable) at zero point will imply that δ is linear and has the form (3) above (resp. $\delta(A) = AT + SA$ for all $A \in \mathrm{Alg}\mathcal{N}$) (Theorem 2.5).

The following notations will be used in this paper.

Let X be a Banach space over the field $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C}$, the field of real numbers or the field of complex numbers), and $B(X)$ denote the algebra of all bounded linear operators on X. Recall that a nest in X is a chain $\mathcal{N}$ of closed (under norm topology) linear subspaces of X containing the trivial subspaces 0 and X, which is closed under the formation of arbitrary closed linear span (denoted by $\bigvee$) and intersection (denoted by $\bigwedge$). $\mathrm{Alg}\mathcal{N}$ denotes the associated nest algebra, which is the algebra of all operators T in $\mathcal{B}(X)$ such that $TN \subseteq N$ for every element $N \in \mathcal{N}$. When $\mathcal{N} \neq \{0, X\}$, we say that $\mathcal{N}$ is nontrivial. It is clear that if $\mathcal{N}$ is trivial, then $\mathrm{Alg}\mathcal{N} = \mathcal{B}(X)$. Denote $\mathrm{Alg}_{\mathcal{F}}\mathcal{N} =: \mathrm{Alg}\mathcal{N} \cap \mathcal{F}(\mathcal{X})$, the set of all finite rank operators in $\mathrm{Alg}\mathcal{N}$. For $N \in \mathcal{N}$, let $N_- = \bigvee\{M \in \mathcal{N} | M \subset N\}$, $N_+ = \bigwedge\{M \in \mathcal{N} | N \subset M\}$ and $N_-^{\perp} = (N_-)^{\perp}$, where $N^{\perp} = \{f \in X^* | N \subseteq \ker(f)\}$ and X^* is the dual of X. Denote $\mathcal{D}(\mathcal{N}) = \bigcup\{N \in \mathcal{N} | N_- \neq X\}$. It is well known that a rank one operator $x \otimes f$ belongs to $\mathrm{Alg}\mathcal{N}$ if and only if there is some $N \in \mathcal{N}$ such that $x \in N$ and $f \in N_-^{\perp}$; every operator in $\mathrm{Alg}_{\mathcal{F}}\mathcal{N}$ is a finite sum of rank one operators in $\mathrm{Alg}_{\mathcal{F}}\mathcal{N}$. Moreover, Erdos in [3] (for Hilbert space case) and Spanoudakis in [13] (for general Banach space case) proved that $\mathrm{Alg}_{\mathcal{F}}\mathcal{N}$ is a dense subset of $\mathrm{Alg}\mathcal{N}$ under the strong operator topology. For more information on nest algebras, we refer to [2].