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Spectral analysis of coupled linear complementarity problems

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ABSTRACT

This note deals with the so-called cone-constrained bivariate eigenvalue problem. The equilibrium model under consideration is a system of linear complementarity problems

$$\begin{cases} P \ni x \perp (Ax + By - \lambda x) \in P^*, \\ Q \ni y \perp (Cx + Dy - \mu y) \in Q^* \end{cases}$$

involving two closed convex cones and their corresponding duals. We study the set of pairs $(\lambda, \mu) \in \mathbb{R}^2$ for which this system has a "nontrivial" solution $(x, y) \in \mathbb{R}^{n+m}$. We discuss also the link between the cone-constrained version and the unconstrained one.

1. Introduction

1.1. The classical setting

The classical bivariate eigenvalue problem consists in finding a pair (λ, μ) of real numbers such that the system of linear equations

$$\begin{cases} Ax + By = \lambda x, \\ Cx + Dy = \mu y \end{cases} \tag{1}$$

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has a solution $(x, y) \in \mathbb{R}^{n+m}$ satisfying the double normalization condition

$$\|x\| = 1, \quad \|y\| = 1.$$

Such a pair $(\lambda, \mu) \in \mathbb{R}^2$ is called a *strong bi-eigenvalue* of the block structured matrix

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The blocks of E are real matrices of appropriate size, namely $A \in \mathbb{M}_n,\ D \in \mathbb{M}_m,\ B \in \mathbb{M}_{n,m}$, and $C \in \mathbb{M}_{m,n}$. If B and C are zero matrices, then the system (1) unfold into two separate standard eigenvalue problems; otherwise, B and C induce a coupling between the state vectors x and y.

Bivariate eigenvalue problems arise in various fields, but surprisingly the theoretical literature on the subject is not so extensive after all. Perhaps the earliest publication introducing a concrete bivariate eigenvalue problem was a 1935 paper by Hotelling [18]. The specific problem treated by Hotelling concerns the determination of canonical correlation coefficients for bivariate statistics; see also [17,22]. An iterative method for solving bivariate eigenvalue problems was proposed in 1961 by Horst [16]. The convergence of Horst's iterative method was proved three decades later by Chu and Watterson [9]. Among a few recent contributions to the theory of bivariate eigenvalue problems we cite the papers by Hanafi and Ten Berge [15], Barkmeijer and van Noorden [3], Chu and Zhang [10], and Liu et al. [24].

For notational simplicity we stick to the bivariate case, but several of our results can be formulated in a multivariate setting. The p-variate version of (1) consists in finding a p-tuple $(\lambda_1, \ldots, \lambda_p)$ of real numbers such that

$$\begin{bmatrix} K_{1,1} & K_{1,2} & \cdots & K_{1,p} \\ K_{2,1} & K_{2,2} & \cdots & K_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ K_{p,1} & K_{p,2} & \cdots & K_{p,p} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_p \end{bmatrix} = \begin{bmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \\ \vdots \\ \lambda_p z_p \end{bmatrix}$$

has a solution $(z_1, z_2, \ldots, z_p) \in \mathbb{R}^{n_1 + \cdots + n_p}$ satisfying the p -fold normalization condition

$$\|z_1\| = 1,\ \|z_2\| = 1,\ \ldots, \|z_p\| = 1.$$

Remark 1.1. Despite an almost identical name, the bivariate eigenvalue problem (1) is fundamentally different from the so-called two-parameter eigenvalue problem. The later one consists in finding a pair $(\lambda, \mu) \in \mathbb{R}^2$ such that

$$\begin{cases} T_1 x = \lambda R_1 x + \mu S_1 x, \\ T_2 y = \lambda R_2 y + \mu S_2 y \end{cases}$$

has a nontrivial solution $(x, y) \in \mathbb{R}^{n+m}$. General information on the two-parameter model can be found in [8,26] and references therein.

1.2. The cone-constrained version

The bivariate eigenvalue problem addressed in this paper is a generalization of (1). In our work, the state vector x is further restricted by means of a nonzero closed convex cone P in $\mathbb{R}^n$. Roughly speaking, P serves to model a possibly infinite number of linear inequality constraints. Similarly, the state vector y is further restricted by means of a nonzero closed convex cone Q in $\mathbb{R}^m$. Instead of a system of linear equations like in (1), the equilibrium model under consideration is now a system of linear complementarity problems:

$$\begin{cases} P \ni x \perp (Ax + By - \lambda x) \in P^*, \\ Q \ni y \perp (Cx + Dy - \mu y) \in Q^*. \end{cases} \tag{2}$$

As usual, the symbol "$\perp$" indicates orthogonality in the appropriate Euclidean space. For instance, $u \perp x$ if and only if $u^T x = 0$ with "T" denoting transposition. The sets