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A difference counterpart to a matrix Hölder inequality

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A B S T R A C T

We point out a sharp reverse Cauchy–Schwarz/Hölder matrix inequality. The Cauchy–Schwarz version involves the usual matrix geometric mean: Let A_i and B_i be positive definite matrices such that $0 < mA_i \leqslant B_i \leqslant MA_i$ for some scalars $0 < m \leqslant M$ and $i = 1, 2, \ldots, n$. Then

$$\left(\sum_{i=1}^n A_i\right) \sharp \left(\sum_{i=1}^n B_i\right) - \sum_{i=1}^n A_i \sharp B_i \leqslant \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{4\left(\sqrt{M}+\sqrt{m}\right)} \sum_{i=1}^n A_i,$$

where the matrix geometric mean of positive definite A and B is defined by

$$A \sharp B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}.$$

1. Introduction

Let a_i and b_i be positive real numbers, $i = 1, 2, \ldots, n$. The Hölder inequality says that

$$\sum_{i=1}^n a_i^{\frac{1}{p}} b_i^{\frac{1}{q}} \leqslant \left(\sum_{i=1}^n a_i\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i\right)^{\frac{1}{q}} \tag{1}$$

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for $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. When $p = q = 2$ in (1), the Cauchy–Schwarz inequality holds. These inequalities can be extended to matrices. Let A and B be positive definite matrices. Their geometric mean is

$$A \sharp B := A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^{1/2} A^{1/2}$$

and a matrix Cauchy–Schwarz inequality for positive definite matrices $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ is:

$$\sum_{i=1}^n A_i \sharp B_i \leqslant \left(\sum_{i=1}^n A_i\right) \sharp \left(\sum_{i=1}^n B_i\right), \tag{2}$$

also see [6]. In a recent paper [7], Lee obtained a sharp reverse inequality for (2):

Theorem A. *Let A_i and B_i be positive definite matrices such that $mA_i \leqslant B_i \leqslant MA_i$ for some scalars $0 < m \leqslant M$ and $i = 1, 2, \ldots, n$. Then*

$$\left(\sum_{i=1}^n A_i\right) \sharp \left(\sum_{i=1}^n B_i\right) \leqslant \frac{\sqrt{M} + \sqrt{m}}{2\sqrt[4]{mM}} \sum_{i=1}^n A_i \sharp B_i.$$

A key feature of this statement is the "sandwich assumption" $mA_i \leqslant B_i \leqslant MA_i$. This leads to more general/precise estimates than simple data of bounds for spectra usually found in the reverse literature, like $\sigma(A_i) \subset [r, s]$ and $\sigma(B_i) \subset [r', s']$ for some $r, s, r', s' > 0$.

Very recently, in order to obtain a reverse matrix Hölder inequality, Bourin et al. [2] extended Theorem A to weighted geometric means. For $\alpha \in [0, 1]$, the weighted geometric mean $A \sharp_\alpha B$ of two positive definite matrices A and B is defined by

$$A \sharp_\alpha B := A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^{\alpha} A^{1/2},$$

so that $A \sharp_\alpha B = A^{1-\alpha} B^\alpha$ whenever A and B commute. The following inequality, for positive definite matrices $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$, is a matrix version for (1)

$$\sum_{i=1}^n A_i \sharp_\alpha B_i \leqslant \left(\sum_{i=1}^n A_i\right) \sharp_\alpha \left(\sum_{i=1}^n B_i\right)$$

and the main result in [2] is a *ratio* type reverse statement. In this short note we complete it by a *difference* type reverse statement.

2. Reverse Cauchy–Schwarz inequality

We start with a difference type reverse of the matrix Cauchy–Schwarz inequality:

Theorem 1. *Let A_i, B_i be positive definite matrices such that $mA_i \leqslant B_i \leqslant MA_i$ for some scalars $0 < m \leqslant M$ and $i = 1, 2, \ldots, n$. Then*

$$\left(\sum_{i=1}^n A_i\right) \sharp \left(\sum_{i=1}^n B_i\right) - \sum_{i=1}^n A_i \sharp B_i \leqslant \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{4\left(\sqrt{M} + \sqrt{m}\right)} \sum_{i=1}^n A_i.$$

To prove it, we need a well-known lemma. The proof given here is adapted from [1].