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Upper and lower bounds on norms of functions of matrices ☆

Anne Greenbaum

University of Washington, Mathematics Department, Box 354350, Seattle, WA 98195, United States

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Abstract

Given an n by n matrix A, we look for a set S in the complex plane and positive scalars m and M such that for all functions p bounded and analytic on S and throughout a neighborhood of each eigenvalue of A, the inequalities

$$m \cdot \inf\{\|f\|_{\mathscr{L}^\infty(S)} \colon f(A) = p(A)\} \leqslant \|p(A)\| \leqslant M \cdot \inf\{\|f\|_{\mathscr{L}^\infty(S)} \colon f(A) = p(A)\}$$

hold. We show that for 2 by 2 matrices, if S is the field of values, then one can take $m = 1$ and $M = 2$. We show that for a perturbed Jordan block – a matrix A that is an n by n Jordan block with eigenvalue 0 except that its $(n, 1)$-entry is ν, with $|\nu| \in (0, 1)$ – if S is the unit disk, then $m = M = 1$. We argue, however, that, in general, due to the behavior of minimal-norm interpolating functions, it may be very difficult or impossible to find such a set S for which the ratio M/m is of moderate size.

Keywords: Pick-Nevanlinna interpolation; Blaschke product; Nonnormal matrix; Field of values; Polynomial numerical hull

1. Introduction

In recent years there has been considerable interest in finding sets in the complex plane that can be associated with a given square matrix or bounded linear operator A to give more information than the spectrum alone can provide about the norms of functions of A. Examples include the *field*

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E-mail address: greenbau@math.washington.edu

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of values or *numerical range* [14,3,4], the ϵ-*pseudospectrum* [19], and the *polynomial numerical hull* of a given degree [16,17,11]. Let S be a set that contains the spectrum of A. One might look for a scalar M (which might or might not depend on A) such that for all functions $p \in \mathscr{H}^\infty(S)$, the Hardy space of bounded analytic functions on S with norm $\|p\|_{\mathscr{L}^\infty(S)} \equiv \sup_{z \in S} |p(z)|$ (and with the additional requirement that p be analytic in a neighborhood of each eigenvalue of A so that $p(A)$ is well defined), the upper bound

$$\|p(A)\| \leqslant M \cdot \|p\|_{\mathscr{L}^\infty(S)} \tag{1}$$

holds.

In this paper, we restrict our attention to n by n matrices A and let $\|\cdot\|$ denote the 2-norm for vectors and the corresponding spectral norm for matrices: $\|A\| \equiv \sup_{\|\mathbf{v}\|=1} \|A\mathbf{v}\|$. We look for sets S where (1) holds and where there is a similar lower bound on $\|p(A)\|$ involving a positive scalar m.

One's first thought might be to look for a positive scalar m such that for all $p \in \mathscr{H}^\infty(S)$,

$$m \cdot \|p\|_{\mathscr{L}^\infty(S)} \leqslant \|p(A)\|. \tag{2}$$

If S is the spectrum of A, and if A is diagonalizable – $A = V\Lambda V^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix of eigenvalues and V a matrix whose columns are eigenvectors – then the following inequalities hold:

$$\|p\|_{\mathscr{L}^\infty(S)} \leqslant \|p(A)\| \leqslant \kappa(V) \cdot \|p\|_{\mathscr{L}^\infty(S)}, \quad \kappa(V) \equiv \|V\| \cdot \|V^{-1}\|. \tag{3}$$

Hence the scalars m and M in (2) and (1) can be taken to be 1 and $\kappa(V)$, respectively. The scalar $m = 1$ is best possible since, for example, if $p(z) \equiv 1$, then $p(A) = I$ and $\|I\| = 1$. If the columns of V are taken to have norm 1 and if the eigenvalues of A are distinct, then the scalar $M = \kappa(V)$ is within a factor of n of optimal, since if $p(\lambda_J) = 1$, where J is the index of a row of V^{-1} with maximal norm and $p(\lambda_i) = 0$ for $i \neq J$, then

$$\|p(A)\| = \|Vp(\Lambda)V^{-1}\| = \|V(:, J)V^{-1}(J, :)\| = \|V^{-1}(J, :)\|,$$

where $V(:, J)$ denotes the Jth column of V and $V^{-1}(J, :)$ denotes the Jth row of V^{-1}, while

$$\kappa(V) \leqslant \|V\|_F \cdot \|V^{-1}\|_F = \sqrt{n} \cdot \left(\sum_{j=1}^{n} \|V^{-1}(j, :)\|^2 \right)^{1/2} \leqslant n \cdot \|V^{-1}(J, :)\|,$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

It is easy to see, however, that if the set S contains points outside the spectrum of A, then there is no positive scalar m for which (2) holds, since if p is the minimal polynomial of A, then $p(A) = 0$ but $p(z) = 0$ only if z is an eigenvalue of A.

One way to circumvent this difficulty is to note that if the degree of the minimal polynomial of A is r, then any function $p(A)$ can be written as a polynomial of degree at most $r - 1$ in A: $p(A) = p_{r-1}(A)$, where p_{r-1} is the polynomial of degree at most $r - 1$ that matches p at the eigenvalues of A and whose derivatives of orders up through $t - 1$ also match those of p at eigenvalues corresponding to a t by t Jordan block. Hence one might look for a set S and a positive scalar m such that for all $p \in \mathscr{H}^\infty(S)$

$$m \cdot \|p_{r-1}\|_{\mathscr{L}^\infty(S)} \leqslant \|p(A)\|, \tag{4}$$

where p_{r-1} is the polynomial of degree at most $r - 1$ satisfying $p_{r-1}(A) = p(A)$. The largest set S for which (4) holds with $m = 1$ is, by definition, the *polynomial numerical hull of degree* $r - 1$ [16,17,11].