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Totally nonnegative (0, 1)-matrices

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We investigate (0, 1)-matrices which are totally nonnegative and therefore which have all of their eigenvalues equal to nonnegative real numbers. Such matrices are characterized by four forbidden submatrices (of orders 2 and 3). We show that the maximum number of 0s in an irreducible, totally nonnegative (0, 1)-matrix of order n is $(n-1)^2$ and characterize those matrices with this number of 0s. We also show that the minimum Perron value of an irreducible, totally nonnegative (0, 1)-matrix of order n equals $2+2\cos\left(\frac{2\pi}{n+2}\right)$ and characterize those matrices with this Perron value.

1. Introduction

Using a trace argument, McKay et al. [4] obtained a result which was the starting point of our investigations and which we formulate as follows.

Theorem 1.1. *Let A be a (0, 1)-matrix of order n each of whose eigenvalues is positive. Then there is a permutation matrix P such that $PAP^t = I_n + B$ where B is a (0, 1)-matrix with 0s on and above the main diagonal. In particular, the eigenvalues of A all equal 1.*

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As formulated in [4], Theorem 1.1 asserts that a digraph D each of whose eigenvalues is positive has a loop at each vertex and does not have any cycles of length strictly greater than 1. In Theorem 1.1, the matrix A is the adjacency matrix of D; the matrix B is the adjacency matrix of an acyclic digraph.

As a corollary of Theorem 1.1 we get the following result.

Corollary 1.2. *Let A be an irreducible $(0, 1)$-matrix of order $n \geqslant 2$ each of whose eigenvalues is nonnegative. Then 0 is an eigenvalue of A and hence A is a singular matrix.*

Proof. If all eigenvalues of A are positive, then by Theorem 1.1, there is a permutation matrix P such that PAP^t is triangular, and hence A is not irreducible if $n \geqslant 2$. Thus 0 is an eigenvalue of A and A is singular. □

Since the trace of a $(0, 1)$-matrix of order n is at most equal to n, the following theorem generalizes Theorem 1.1.

Theorem 1.3. *Let A be a $(0, 1)$-matrix of order n with trace at most r and with r positive eigenvalues and $n - r$ zero eigenvalues. Then there is a permutation matrix P such that $PAP^T = D + B$ where B is a $(0, 1)$-matrix with 0s on and above the main diagonal and D is a $(0, 1)$-diagonal matrix with r 1s. In particular, A has r eigenvalues equal to 1, $n - r$ eigenvalues equal to 0, and the trace of A equals r.*

Proof. The proof starts by using the technique of [4]. Let the eigenvalues of A be

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_n.$$

Using the arithmetic/geometric mean inequality, we have

$$1 \geqslant \frac{\text{trace}(A)}{r} = \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_r}{r} \geqslant (\lambda_1\lambda_2\cdots\lambda_r)^{1/r}. \tag{1}$$

The sum α_r of the determinants of the principal submatrices of order r of A equals the sum of the products of the eigenvalues of A taken r at a time and so equals $\lambda_1\lambda_2\cdots\lambda_r$ and is positive. Since A is an integral matrix, α_r is an integer and thus $\alpha_r \geqslant 1$. Thus using (1) we get

$$1 \geqslant \frac{\text{trace}(A)}{r} = \frac{\lambda_1 + \lambda_2 \cdots + \lambda_r}{r} \geqslant (\lambda_1\lambda_2\cdots\lambda_r)^{1/r} \geqslant 1. \tag{2}$$

Hence we have equality throughout in (2). This implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_r$, and this common value equals 1. Thus A has r eigenvalues equal to 1, and $n - r$ eigenvalues equal to 0, and the trace of A equals r. Since A is a nonnegative matrix, it follows from the classical Perron–Frobenius theory that A has r irreducible components $A_1, A_2, \ldots, A_r$ each of which has spectral radius (maximum eigenvalue) 1, and all other eigenvalues equal to 0; the remaining irreducible components, if any, are zero matrices of order 1. Since each A_i is irreducible, each A_i has at least one 1 in each row and column. Again by the Perron–Frobenius theory, each A_i is a permutation matrix corresponding to a permutation cycle. Since the eigenvalues of A_i are one 1 and then all 0s, we conclude that each A_i has order 1. Thus A has r 1s and $n - r$ 0s on the main diagonal, and all 0s above the main diagonal. □

Notice that again we conclude that the digraph whose adjacency matrix is A does not have any cycles of length strictly greater than 1.

From Theorems 1.1 and 1.3, we conclude that if A is a (0,1)-matrix of order n with either

(i) n positive eigenvalues (the trace of A then equals n by Theorem 1.1), or
(ii) $n - 1$ positive eigenvalues, one zero eigenvalue, and trace equal to (or at most equal to) $n - 1$, then A is simultaneously permutable to a triangular matrix. Using the arithmetic/geometric mean inequality as in the proof of Theorem 1.3, we see that if A has $n - 1$ positive eigenvalues and one zero eigenvalue, then the trace of A is $n - 1$ or n. If in (ii) we replace trace equal to $n - 1$ with trace equal to n, then A need not be simultaneously permutable to a triangular matrix. For example, the irreducible matrix