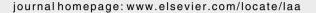


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## Linear Algebra and its Applications





## Totally nonnegative (0, 1)-matrices

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#### ARTICLE INFO

#### Article history: Received 27 April 2009 Accepted 18 November 2009 Available online 14 December 2009

Submitted by L. Hogben

AMS classification:

05C50

15A18

15A42 15A48

. .

Keywords: Totally nonnegative matrices Digraphs Spectrum

Eigenvalues

(0, 1)-Matrices

#### ABSTRACT

We investigate (0,1)-matrices which are totally nonnegative and therefore which have all of their eigenvalues equal to nonnegative real numbers. Such matrices are characterized by four forbidden submatrices (of orders 2 and 3). We show that the maximum number of 0s in an irreducible, totally nonnegative (0,1)-matrix of order n is  $(n-1)^2$  and characterize those matrices with this number of 0s. We also show that the minimum Perron value of an irreducible, totally nonnegative (0,1)-matrix of order n equals n0, and characterize those matrices with this Perron value.

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#### 1. Introduction

Using a trace argument, McKay et al. [4] obtained a result which was the starting point of our investigations and which we formulate as follows.

**Theorem 1.1.** Let A be a (0, 1)-matrix of order n each of whose eigenvalues is positive. Then there is a permutation matrix P such that  $PAP^t = I_n + B$  where B is a (0, 1)-matrix with 0s on and above the main diagonal. In particular, the eigenvalues of A all equal 1.

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As formulated in [4], Theorem 1.1 asserts that a digraph D each of whose eigenvalues is positive has a loop at each vertex and does not have any cycles of length strictly greater than 1. In Theorem 1.1, the matrix A is the adjacency matrix of D; the matrix B is the adjacency matrix of an acyclic digraph.

As a corollary of Theorem 1.1 we get the following result.

**Corollary 1.2.** Let A be an irreducible (0, 1)-matrix of order  $n \ge 2$  each of whose eigenvalues is nonnegative. Then 0 is an eigenvalue of A and hence A is a singular matrix.

**Proof.** If all eigenvalues of *A* are positive, then by Theorem 1.1, there is a permutation matrix *P* such that  $PAP^t$  is triangular, and hence *A* is not irreducible if  $n \ge 2$ . Thus 0 is an eigenvalue of *A* and *A* is singular.  $\square$ 

Since the trace of a (0, 1)-matrix of order n is at most equal to n, the following theorem generalizes Theorem 1.1.

**Theorem 1.3.** Let A be a (0, 1)-matrix of order n with trace at most r and with r positive eigenvalues and n-r zero eigenvalues. Then there is a permutation matrix P such that  $PAP^T = D + B$  where B is a (0, 1)-matrix with  $PAP^T = D + B$  where  $PAP^T = D + B$  wher

**Proof.** The proof starts by using the technique of [4]. Let the eigenvalues of *A* be

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_n$$
.

Using the arithmetic/geometric mean inequality, we have

$$1 \geqslant \frac{\operatorname{trace}(A)}{r} = \frac{\lambda_1 + \lambda_2 + \dots + \lambda_r}{r} \geqslant (\lambda_1 \lambda_2 \dots \lambda_r)^{1/r}. \tag{1}$$

The sum  $\alpha_r$  of the determinants of the principal submatrices of order r of A equals the sum of the products of the eigenvalues of A taken r at a time and so equals  $\lambda_1 \lambda_2 \cdots \lambda_r$  and is positive. Since A is an integral matrix,  $\alpha_r$  is an integer and thus  $\alpha_r \ge 1$ . Thus using (1) we get

$$1 \geqslant \frac{\operatorname{trace}(A)}{r} = \frac{\lambda_1 + \lambda_2 \cdots + \lambda_r}{r} \geqslant (\lambda_1 \lambda_2 \cdots \lambda_r)^{1/r} \geqslant 1.$$
 (2)

Hence we have equality throughout in (2). This implies that  $\lambda_1 = \lambda_2 = \cdots = \lambda_r$ , and this common value equals 1. Thus A has r eigenvalues equal to 1, and n-r eigenvalues equal to 0, and the trace of A equals r. Since A is a nonnegative matrix, it follows from the classical Perron–Frobenius theory that A has r irreducible components  $A_1, A_2, \ldots, A_r$  each of which has spectral radius (maximum eigenvalue) 1, and all other eigenvalues equal to 0; the remaining irreducible components, if any, are zero matrices of order 1. Since each  $A_i$  is irreducible, each  $A_i$  has at least one 1 in each row and column. Again by the Perron–Frobenius theory, each  $A_i$  is a permutation matrix corresponding to a permutation cycle. Since the eigenvalues of  $A_i$  are one 1 and then all 0s, we conclude that each  $A_i$  has order 1. Thus A has r 1s and n-r 0s on the main diagonal, and all 0s above the main diagonal.

Notice that again we conclude that the digraph whose adjacency matrix is *A* does not have any cycles of length strictly greater than 1.

From Theorems 1.1 and 1.3, we conclude that if A is a (0,1)-matrix of order n with either

- (i) *n* positive eigenvalues (the trace of *A* then equals *n* by Theorem 1.1), or
- (ii) n-1 positive eigenvalues, one zero eigenvalue, and trace equal to (or at most equal to) n-1, then A is simultaneously permutable to a triangular matrix. Using the arithmetic/geometric mean inequality as in the proof of Theorem 1.3, we see that if A has n-1 positive eigenvalues and one zero eigenvalue, then the trace of A is n-1 or n. If in (ii) we replace trace equal to n-1 with trace equal to n, then n need not be simultaneously permutable to a triangular matrix. For example, the irreducible matrix

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