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Idempotent matrices over antirings

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ABSTRACT

We study the idempotent matrices over a commutative antiring. We give a characterization of idempotent matrices by digraphs. We study the orbits of conjugate action and find the cardinality of orbits of basic idempotents. Finally, we prove that invertible, linear and idempotent preserving operators on $n \times n$ matrices over entire antirings are exactly conjugate actions for $n \geqslant 3$. We also give a complete characterization of the 2×2 case.

1. Introduction

A *semiring* is a set S equipped with binary operations $+$ and $\cdot$ such that $(S, +)$ is a commutative monoid with identity element 0 and $(S, \cdot)$ is a monoid with identity element 1. In addition, operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates S. A semiring is *commutative* if $ab = ba$ for all $a, b \in S$.

A semiring S is called an *antiring* if it is zerosumfree, i.e., if the condition $a + b = 0$ implies that $a = b = 0$ for all $a, b \in S$. Note that some authors also refer to antirings as *antinegative* semirings.

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An antiring is called *entire* if $ab = 0$ implies that either $a = 0$ or $b = 0$.

An antiring S is said to be *cancellative* if the equation $a + b = a + c$ implies that $b = c$ for all $a, b, c \in S$.

For example, the set of nonnegative integers with the usual operations of addition and multiplication is a commutative entire antiring. Inclines (which are additively idempotent semirings in which products are less than or equal to either factor) are commutative antirings. Distributive lattices are inclines and thus antirings. Also Boolean algebras are commutative antirings.

If R is an ordered ring and P its positive cone, then $R = -P \cup \{0\} \cup P$ and $R^+ = P \cup \{0\}$ is a cancellative antiring. Therefore, $\mathbb{R}^+$ and $\mathbb{Z}^+$ are both cancellative antirings.

Let us denote by $U(S)$ the group of all invertible elements in S, i.e., $U(S) = \{a \in S;\, ab = ba = 1 \text{ for some } b \in S\}$.

Let $M_n(S)$ denote the set of all $n \times n$ matrices with entries from S. We denote by $A(i,j)$ the entry in the ith row and jth column of matrix A.

In this paper, we study the idempotent matrices over a commutative antiring. Idempotent matrices over Boolean algebras, chain semirings and inclines were studied for example in [2,3,5]. Here, we generalize some of their results.

First, we characterize the set of directed graphs of idempotent matrices over antirings, thus finding the possible patterns of idempotent matrices. Next, we study the orbits of conjugate action and find the cardinality of orbits for the basic types of idempotents. This is important because in Section 4, we prove that idempotent matrix preservers over entire antirings are exactly conjugations. We characterize linear, invertible and idempotent preserving operators, and thus improve upon the result given in [5]. Namely, we prove the following theorems.

Theorem. *Let S be an entire antiring and $n \geqslant 3$. An invertible linear operator T on $M_n(S)$ preserves idempotents if and only if there exists $U \in GL_n(S)$ such that either*

$$T(X) = UXU^{-1} \quad \text{for all } X \in M_n(S)$$

or

$$T(X) = UX^TU^{-1} \quad \text{for all } X \in M_n(S).$$

However, this is not true for invertible linear idempotent preservers on 2×2 matrices. We say, that an idempotent $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is strongly connected if $b \neq 0$ and $c \neq 0$.

Theorem. *Let S be an entire cancellative antiring and $U(S) \neq \{1\}$. There exists a 2×2 strongly connected idempotent matrix over S if and only if for every invertible, linear and idempotent preserving operator T on $M_2(S)$ there exist $0 \neq \alpha \in S$ and $U \in GL_2(S)$ such that either*

$$T(\alpha X) = U(\alpha X)U^{-1} \quad \text{for all } X \in M_2(S)$$

or

$$T(\alpha X) = U(\alpha X)^TU^{-1} \quad \text{for all } X \in M_2(S).$$

Moreover, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an idempotent, such that $b, c \in U(S)$, then for every invertible, linear and idempotent preserving operator T on $M_2(S)$ there exists $U \in GL_2(S)$ such that either

$$T(X) = UXU^{-1} \quad \text{for all } X \in M_2(S)$$

or

$$T(X) = UX^TU^{-1} \quad \text{for all } X \in M_2(S).$$

Note that recently, the results in [5] were generalized to non-invertible idempotent preserving linear operators over certain classes of antirings in [4].

In this paper we assume that all antirings are commutative.