Linear Algebra and its Applications 431 (2009) 855–862

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Interval systems over idempotent semiring

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ARTICLE INFO

Article history:
Received 18 September 2008
Accepted 25 March 2009
Available online 5 May 2009

Submitted by P. Butkovic

AMS classification:
Primary: 65G40
Secondary: 06F05, 15A09

Keywords:
Max algebra
Idempotent semiring
Interval analysis
Residuation theory

ABSTRACT

This paper deals with solution of inequality $\mathbf{A} \otimes \mathbf{x} \preceq \mathbf{b}$, where $\mathbf{A}$, $\mathbf{x}$ and $\mathbf{b}$ are interval matrices with entries defined over idempotent semiring. It deals also with the computation of a pair of intervals, $(\mathbf{x}, \mathbf{y})$ which satisfies the equation $\mathbf{A} \otimes \mathbf{x} = \mathbf{B} \otimes \mathbf{y}$. It will be shown that this equation may be solved by considering the interval version of the iterative scheme proposed in [R.A. Cuninghame-Green, P. Butkovič, The equation $ax = by$ over $(max, +)$, Theoret. Comput. Sci. 293 (2003) 3–12].

1. Introduction

Many problems in the optimization theory and other fields of mathematics are non-linear in the traditional sense but appear to be linear over idempotent semirings (e.g., see [1,3,6,10]). Idempotency of the additive law induces that idempotent semirings are (partially) ordered sets. The Residuation theory [2,5,8] is a suitable tool to deal with inverse problems of order preserving mappings. It is usually used to solve equations defined over idempotent semiring [1,6,7], for instance the greatest solution of inequality $Ax \preceq b$ may be computed by means of this theory.

Interval mathematics was pioneered by Moore (see [16]) as a tool for bounding rounding errors in computer programs. Since then, interval mathematics has been developed into a general methodology for investigating numerical uncertainty in numerous problems and algorithms. In [14] the idempotent version is addressed. The authors show that idempotent interval mathematics appears to

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.03.039

be remarkably simpler than its traditional analog. For example, in the traditional interval arithmetic, multiplication of intervals is not distributive with respect to addition of intervals, while idempotent interval arithmetic keeps this distributivity. This paper deals first with solution of inequality $\mathbf{A} \otimes \mathbf{x} \preceq \mathbf{b}$, where $\mathbf{A}$, $\mathbf{x}$ and $\mathbf{b}$ are interval matrices (see Proposition 31). When equality is achieved, according to definition given in [4], the equations system is said weakly solvable since at least one of its subsystems is solvable. In a second step, the paper deals with the computation of a pair of intervals, $(\mathbf{x}, \mathbf{y})$ which satisfies the equation $\mathbf{A} \otimes \mathbf{x} = \mathbf{B} \otimes \mathbf{y}$. It will be shown that this equation may be solved by considering the interval version of the iterative scheme proposed in [7].

2. Preliminaries

Definition 1. A *semiring* $\mathcal{S}$ is a set endowed with two internal operations denoted by $\oplus$ (addition) and $\otimes$ (multiplication), both associative and both having neutral elements denoted by ε and e, respectively, such that $\oplus$ is also commutative and idempotent (*i.e.* $a \oplus a = a$). The $\otimes$ operation is distributive with respect to $\oplus$, and ε is absorbing for the product (*i.e.* $\forall a, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). When $\otimes$ is commutative, the semiring is said to be commutative.

Semirings can be endowed with a canonical *order* defined by: $a \succeq b$ iff $a = a \oplus b$. Then they become sup-semilattices and $a \oplus b$ is the least upper bound of a and b. A semiring is *complete* if sums of infinite number of terms are always defined, and if multiplication distributes over infinite sums too. In particular, the sum of all elements of a complete semiring is defined and denoted by $\top$ (for 'top'). A complete semiring (sup-semilattice) becomes a complete lattice for which the greatest lower bound of a and b is denoted $a \wedge b$, *i.e.*, the least upper bound of the (nonempty) subset of all elements which are less than a and b (see [1, Section 4]).

Example 2 ((*max*, +) *algebra*). The set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the max operator as $\oplus$ and the classical sum as $\otimes$ is a complete idempotent semiring of which $\varepsilon = -\infty$, $e = 0$ and $\top = +\infty$ and the greatest lower bound $a \wedge b = min(a, b)$.

Example 3 ((*max*, *min*) *algebra*). The set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the max operator as $\oplus$ and the min operator as $\otimes$ is a complete idempotent semiring of which $\varepsilon = -\infty$, $e = +\infty$ and $\top = +\infty$ and the greatest lower bound $a \wedge b = min(a, b)$.

Definition 4 (*Subsemiring*). A subset $\mathcal{C}$ of a semiring is called a subsemiring of $\mathcal{S}$ if

- $\varepsilon \in \mathcal{C}$ and $e \in \mathcal{C}$;
- $\mathcal{C}$ is closed for $\oplus$ and $\otimes$, i.e, $\forall a, b \in \mathcal{C}, a \oplus b \in \mathcal{C}$ and $a \otimes b \in \mathcal{C}$.

Definition 5 (*Principal order ideal*). Let $\mathcal{S}$ be an idempotent semiring. An order ideal set is a nonempty subset $\mathcal{X}$ of $\mathcal{S}$ such that $(x \in \mathcal{X}$ and $y \preceq x) \Rightarrow y \in \mathcal{X}$. A principal order ideal (generated by x) is an order ideal, denoted $_{\downarrow}\mathcal{X}_x$, of the form $_{\downarrow}\mathcal{X}_x := \{y \in \mathcal{S} | y \preceq x\}$.

The residuation theory provides, under some assumptions, *greatest* solutions to inequalities such as $f(x) \preceq b$ where f is an order preserving mapping (*i.e.*, $a \preceq b \Rightarrow f(a) \preceq f(b)$) defined over ordered sets.

Definition 6 (*Residual and residuated mapping*). An order preserving mapping $f : \mathcal{D} \to \mathcal{E}$, where $\mathcal{D}$ and $\mathcal{E}$ are ordered sets, is a *residuated mapping* if for all $y \in \mathcal{E}$, the least upper bound of the subset $\{x | f(x) \preceq y\}$ exists and belongs to this subset. It is then denoted by $f^\sharp(y)$. Mapping $f^\sharp$ is called the residual of f. When f is residuated, $f^\sharp$ is the unique order preserving mapping such that

$$f \circ f^\sharp \preceq \mathsf{Id}_{\mathcal{E}} \quad \text{and} \quad f^\sharp \circ f \succeq \mathsf{Id}_{\mathcal{D}}, \tag{1}$$

where Id is the identity mapping respectively on $\mathcal{D}$ and $\mathcal{E}$.