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On two inequalities for the Hadamard product and the Fan product of matrices ☆

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If A and B are $n \times n$ nonsingular M-matrices, a lower bound on the smallest eigenvalue $\tau(A\bigstar B)$ for the Fan product of A and B is given. In addition, using the estimate on the perron root of nonnegative matrices, we also obtain an upper bound on the spectral radius $\rho(A \circ B)$ for nonnegative matrices A and B. These bounds improve some existing results.

1. Introduction

For a positive integer n, N denotes the set $\{1, 2, \ldots, n\}$. The set of all $n \times n$ complex matrices is denoted by $C^{n\times n}$ and $R^{n\times n}$ denotes the set of all $n \times n$ real matrices throughout.

Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two real $n \times n$ matrices. Then, $A \geqslant B(> B)$ if $a_{i,j} \geqslant b_{i,j}(> b_{i,j})$ for all $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n$. If O is the null matrix and $A \geqslant O(> O)$, we say that A is a nonnegative (positive)

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matrix. The spectral radius of A is denoted by $\rho(A)$. If A is a nonnegative matrix, the Perron–Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$, where $\sigma(A)$ denotes the spectrum of A.

For $n \geqslant 2$, an $n \times n$ $A \in C^{n\times n}$ is reducible if there exists an $n \times n$ permutation matrix P such that

$$P^TAP = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ is an $r \times r$ submatrix and $A_{2,2}$ is an $(n-r) \times (n-r)$ submatrix, where $1 \leqslant r < n$. If no such permutation matrix exists, then A is irreducible. If A is a 1×1 complex matrix, then A is irreducible if its single entry is nonzero, and reducible otherwise.

Let A be an irreducible nonnegative matrix. It is well known that there exists a positive vector u such that $Au = \rho(A)u$, u being called right Perron eigenvector of A.

The *Hadamard product* of $A \in C^{n\times n}$ and $B \in C^{n\times n}$ is defined by $A \circ B \equiv (a_{i,j}b_{i,j}) \in C^{n\times n}$.

In [3, p. 358], there is a simple estimate for $\rho(A \circ B)$: if $A, B \in R^{n\times n}$, $A \geqslant O$, and $B \geqslant O$, then $\rho(A \circ B) \leqslant \rho(A)\rho(B)$. From Exercise [3, p. 358], we know this inequality can be very weak by taking $B = J$, the matrix of all ones. For example, If $A = I$, $B = J$, then we have

$$\rho(A \circ B) = \rho(A) = 1 \ll \rho(A)\rho(B) = n$$

when n is very large. But also clearly show that equality can occur (let $A = I$ and $B = I$).

Recently, Fang [4] gave an upper bound for $\rho(A \circ B)$, that is,

$$\rho(A \circ B) \leqslant \max_{1\leqslant i\leqslant n}\{2a_{i,i}b_{i,i} + \rho(A)\rho(B) - a_{i,i}\rho(B) - b_{i,i}\rho(A)\} \tag{1}$$

which is shaper than the bound $\rho(A)\rho(B)$ in [3, p. 358].

For two nonnegative matrices A, B, we will give a new upper bound for $\rho(A \circ B)$ in Section 2. The bound is shaper than the bound $\rho(A)\rho(B)$ in [3, p. 358] and the bound $\max_{1\leqslant i\leqslant n}\{2a_{i,i}b_{i,i} + \rho(A)\rho(B) - a_{i,i}\rho(B) - b_{i,i}\rho(A)\}$ in [4].

The set $Z_n \subset R^{n\times n}$ is defined by

$$Z_n = \{A = (a_{i,j}) \in R^{n\times n} : a_{i,j} \leqslant 0 \quad \text{if } i \neq j,\ i,j = 1,\ldots,n\}$$

the simple sign patten of the matrices in Z_n has many striking consequences. Let $A = (a_{i,j}) \in Z_n$ and suppose $A = \alpha I - P$ with $\alpha \in R$ and $P \geqslant O$. Then $\alpha - \rho(P)$ is an eigenvalue of A, every eigenvalue of A lies in the disc $\{z \in C : |z - \alpha| \leqslant \rho(P)\}$, and hence every eigenvalue λ of A satisfies $Re\lambda \geqslant \alpha - \rho(P)$. In particular, A is an M-matrix if and only if $\alpha > \rho(P)$. If A is an M-matrix, one may always write $A = \gamma I - P$ with $\gamma = \max\{a_{i,i} : i = 1,\ldots,n\}$, $P = \gamma I - A \geqslant O$; necessarily, $\gamma > \rho(P)$.

If $A = (a_{i,j}) \in Z_n$, and if we denote $\min\{Re(\lambda) : \lambda \in \sigma(A)\}$ by $\tau(A)$. Basic for our purpose are the following simple facts (see Problem 16, 19 and 28 in Section 2.5 of [3]):

(i) $\tau(A) \in \sigma(A)$; $\tau(A)$ is called the minimum eigenvalue of A.
(ii) If $A, B \in Z_n$, and $A \geqslant B$, then $\tau(A) \geqslant \tau(B)$.
(iii) If $A \in Z_n$, then $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}, and $\tau(A) = \frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of A.

Let A be an irreducible nonsingular M-matrix. It is well known that there exists a positive vector u such that $Au = \tau(A)u$, u being called right Perron eigenvector of A.

Let $A \in C^{n\times n}$, $B \in C^{n\times n}$. The *Fan product* of A and B is denoted by $A \bigstar B \equiv C = (c_{i,j}) \in C^{n\times n}$ and is defined by

$$c_{i,j} = \begin{cases} -a_{i,j}b_{i,j}, & \text{if } i \neq j, \\ a_{i,i}b_{i,i}, & \text{if } i = j. \end{cases}$$

If $A, B \in Z_n$ are M-matrices, then so is $A \bigstar B$. In [3, p. 359], a lower bound for $\tau(A \bigstar B)$ was given: Let $A, B \in Z_n$ be M-matrices. Then $A^{-1} \circ B^{-1} \geqslant (A \bigstar B)^{-1}$, and hence $\tau(A \bigstar B) \geqslant \tau(A)\tau(B)$. Fang [4] gave a sharper lower bound for $\tau(A \bigstar B)$, that is,

$$\tau(A \bigstar B) \geqslant \min_{1\leqslant i\leqslant n}\{a_{i,i}\tau(B) + b_{i,i}\tau(A) - \tau(A)\tau(B)\}. \tag{2}$$