Linear Algebra and its Applications 431 (2009) 985–990

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Factorization in noncommutative curves

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ARTICLE INFO

Article history:
Received 19 November 2007
Accepted 30 March 2009
Available online 5 May 2009

Submitted by R. Guralnick

AMS classification:
14A22
14H50
14R
16D60
16G30

Keywords:
Modules
Plane curves
Noncommutative factorization
Ext-relations

ABSTRACT

A commutative curve $(f_0) \in k[x_1, \ldots, x_n]$ has many noncommutative models, i.e. $f \in k\langle x_1, \ldots, x_n\rangle$ having f_0 as its image by the canonical epimorphism κ from $k\langle x_1, \ldots, x_n\rangle$ to $k[x_1, \ldots, x_n]$.
In this note we consider the cases, where $n = 2$.
If the polymomial f_0 has an irreducible factor, g_0, then in terms of conditions on the noncommutative models of (f_0), we determine, when g_0^2 is a factor of f_0.
In fact we prove that in case there exists a noncommutative model f of f_0 such that $Ext_A^1(P, Q) \neq 0$ for all point $P, Q \in \mathbf{Z}(f_0)$, where $A = k\langle x, y\rangle/(f)$, then g_0^2 is a factor of f_0.
We also note that the "converse" result holds.
Next we apply the methods from above to show that in case an element f in the free algebra has 2 essential different factorizations

$$f = gh = h_1 g' h_2,$$

where $g_0 = g_0'$ and with g_0 irreducible and prime to h_0, then

$$\mathbf{Z}(g_0) \cap \mathbf{Z}((h_1)_0) = \emptyset,$$

i.e. g_0 and $(h_1)_0$ do not have a common zero.

1. Introduction

We consider algebras over an algebraically closed field, k, of characteristic 0. Of particular interest for us here are the cases where the algebras are of the form $k\langle x, y\rangle/(f)$ for more results in this direction see also [3,4].

As usual $k\langle x, y\rangle$ denotes the free algebra on two generators and $0 \neq f \in k\langle x, y\rangle$.

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doi:10.1016/j.laa.2009.03.050

We have the canonical epimorhism κ from $k\langle x,y\rangle$ to the ordinary polynomialring $k[x,y]$. $\kappa(f)$ is denoted by f_0 and we say f is a (noncommutative) model of the curve f_0 (or (f_0)). (Note that a 1-dimensional representation of the algebra can be considered as a point on the "commutative" curve $f_0 = 0$.)

Clearly if one adds an element from the commutator ideal, $([x,y])$, to f, one gets the same commutative curve f_0, i.e. the same 1-dimensional representations.

While for a commutative algebra, R, $Ext^1_R(P,Q) = 0$ for 2 non-isomorphic simple modules P and Q. This is no longer the case for noncommutative algebras. The noncommutative situation is studied in details in [3,4], where [4, Theorem 4] and [4, Theorem 5.2] give particular examples on how knowledge of $Ext^1_A(P,Q)$ for some simple 1-dimensional A-modules P and Q can give some information on the ideal (f), where $A = k\langle x,y\rangle/(f)$.

In case we have $f = gh_1gh_2$, for elements f, g, h_1 and $h_2 \in k\langle x,y\rangle$, it readily follows from our methods that with $A = k\langle x,y\rangle/(f)$, $Ext^1_A(P,Q) \neq 0$ for all points P and Q from $\mathbf{Z}(g_0)$. We prove a sort of converse:

Suppose $f_0 = g_0h_0$ has a noncommutative model f with $Ext^1_A(P,Q) \neq 0$ for all $P, Q \in \mathbf{Z}(g_0)$, then g_0^2 is a factor of f_0.

We apply the methods from above to factorization questions in the free algebra $k\langle x,y\rangle$:

As is well known the factorization

$$x_1x_2x_1 + x_1 = x_1(x_2x_1 + 1) = (x_1x_2 + 1)x_1$$

shows that one does not have unique factorisation in the classical sense in $k\langle x_1,\ldots,x_m\rangle$, but there is a unique factorization theorem [1, Section 3.3].

We prove that in case $f = gh = h_1g'h_2$, where g_0 is reduced and prime to h_0, then $\mathbf{Z}(g_0) \cap \mathbf{Z}((h_1)_0) = \emptyset$.

2. First main result

For the readers convenience we start by recalling the following terminology from [3,4]:

Let $S = k\langle x_1,\ldots,x_m\rangle$ denote the free k-algebra on m noncommuting variables. Let ϕ_P denote the 1-dimensional representation of S corresponding to a point $P = (a_1,\ldots,a_m) \in A^m_k$.

We then get maps $D_i(\ ;P) \in Der_k(S, Hom_k(P,S))$, defined by

$$D_i(a;P) = 0, \quad \text{when } a \in k, \tag{1}$$

$$D_i(x_j;P) = \delta_{ij}, \tag{2}$$

$$D_i(fg;P) = fD_i(g;P) + D_i(f;P)\phi_P(g). \tag{3}$$

The element

$$D_k(f;P)$$

is called the noncommutative k-th partial derivative of f with respect to the 1-dimensional representation of S determined by P and usually we write $g(P)$ in stead of $\phi_P(g)$.

In the situation where $A = k\langle x_1,\ldots,x_m\rangle/I$ and I is generated as a twosided ideal by $f^1,\ldots,f^r$, the left ideal of A generated by the images of the i-th partial derivatives of the generators is denoted by

$$J_i(I, f^1,\ldots,f^r;P),$$

this is independent of the choice of generators for I [3, Lemma 4.5] and one has the following [3, Proposition 4.7]:

Let $A = k\langle x_1,\ldots,x_m\rangle/I$ be a k-algebra and let ϕ_P and ϕ_Q be two 1 dimensional representations of A corresponding to points P and Q. Suppose $P \neq Q$ then

$$\dim_k Ext^1_A(P,Q) = m - 1 - rkJ(I;P)(Q). \tag{4}$$

In case $m = 2$ and $I = (f)$ we get:

Let A denote the algebra $k\langle x,y\rangle/(f)$ and let P and Q be two different points corresponding to 1-dimensional representations of A. Then