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On Ando–Li–Mathias geometric mean equations

Yongdo Lim *

Department of Mathematics, Kyungpook National University, Taegu 702-701, Republic of Korea

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Abstract

In this paper we consider a family of non-linear matrix equations based on the higher-order geometric means of positive definite matrices that proposed by Ando–Li–Mathias. We prove that the geometric mean equation

$$X = B + G(A_1, A_2, \ldots, A_m, \underbrace{X, X, \ldots, X}_{n})$$

has a unique positive definite solution depending continuously on the parameters of positive definite A_i and positive semidefinite B. It is shown that the unique positive definite solutions $G_n(A_1, A_2, \ldots, A_m)$ for $B = 0$ satisfy the minimum properties of geometric means, yielding a sequence of higher-order geometric means of positive definite matrices.

Keywords: Positive definite matrix; Geometric mean; Non-linear matrix equation; Contraction; Fixed point; Thompson metric

1. Introduction

The matrix equation $X = Q - A^*X^{-1}A$ with Q positive definite has been studied recently by several authors (see [1,6–8,10,19–22]). For the application areas in which the equations arise, see the references therein. As a special case, the non-linear matrix equation

$$X = T - BX^{-1}B \tag{1.1}$$

* Tel.: +82 53 950 6834; fax: +82 53 950 6306.
E-mail address: ylim@knu.ac.kr

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where T is positive definite and B is positive semidefinite, is solved by the author [18] via Anderson–Morley–Trapp [1] and Engwerda's results (Theorem 11 of [7,8]): It has a positive definite solution if and only if $2B \leqslant T$, and the maximal and minimal positive definite solutions are explicitly described in terms of geometric mean of positive definite matrices:

$$X_+ = \frac{1}{2}(T + (T + 2B)\#(T - 2B)), \tag{1.2}$$

$$X_- = \frac{1}{2}(T - (T + 2B)\#(T - 2B)), \tag{1.3}$$

respectively, where $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ denotes the geometric mean of positive definite matrices A and B. Realizing the geometric mean $A\#B$ as a unique positive definite solution of the Riccati equation $XA^{-1}X = B$, Eq. (1.1) under the side conditions $T > 2B$ and $X > B$ is equivalent to the following geometric mean equation:

$$X = B + C\#X \quad (T = 2B + C).$$

Recently Ando–Li–Mathias [2] proposed a successful definition of geometric mean $G(A_1, A_2, \ldots, A_n)$ of n-positive definite matrices A_i via symmetrization procedure. The main concern of this paper is the extended geometric mean equations based on the Ando–Li–Mathias's geometric mean of several positive definite matrices:

$$X = B + G(A_1, A_2, \ldots, A_m, \underbrace{X, X, \ldots, X}_{n}) \quad (n = 0, 1, 2, \ldots). \tag{1.4}$$

We show that Eq. (1.4) has a unique positive definite solution depending continuously on the parameters of positive definite A_i and positive semidefinite B. For $B = 0$, the unique positive definite solution, denoted by $G_n(A_1, A_2, \ldots, A_m)$, is viewed as a matrix mean and satisfies all properties of the geometric mean of Ando–Li–Mathias presented in [2]. This provides a sequence of higher-order geometric means of positive definite matrices and yields a problem to distinguish these geometric means with that of Ando–Li–Mathias.

Throughout this paper, we assume that $\Omega = \Omega(k)$ is the convex cone of positive definite $k \times k$ Hermitian matrices. For Hermitian matrices X and Y, we write that $X \leqslant Y$ if $Y - X$ is positive semidefinite, and $X < Y$ if $Y - X$ is positive definite (positive semidefinite and invertible).

2. Higher order geometric mean

Let (X, d) be a metric space. A k-mean on X is a k-ary operation $\mu : X^k \to X$ that satisfies a generalized idempotency law: $\mu(x, \ldots, x) = x$ for all $x \in X$. We need some preliminaries: A k-mean on X is called non-expansive if it satisfies for all $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_k) \in X^k$

$$d(\mu(\mathbf{x}), \mu(\mathbf{y})) \leqslant d_s(\mathbf{x}, \mathbf{y}) := \max_{1 \leqslant j \leqslant k} d(x_j, y_j). \tag{2.5}$$

For $0 < \rho < 1$, a k-mean μ on X is called coordinatewise ρ-contractive if for any $\mathbf{x}, \mathbf{y} \in X^k$ that differ only in one coordinate, say $x_j \neq y_j$,

$$d(\mu(\mathbf{x}), \mu(\mathbf{y})) \leqslant \rho d(x_j, y_j).$$

Moreover, the barycentric operator $\beta : X^{k+1} \to X^{k+1}$ is defined by

$$\beta(\mathbf{x}) = (\mu(\pi_{\neq 1}\mathbf{x}), \ldots, \mu(\pi_{\neq k+1}\mathbf{x})),$$