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ScienceDirect
Linear Algebra and its Applications 418 (2006) 682–692
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

# Correlation matrices of yields and total positivity

Ernesto Salinelli [a], Carlo Sgarra [b,*]

[a] Dipartimento di Scienze Economiche e Metodi Quantitativi, Università del Piemonte Orientale "A. Avogadro", Via Perrone 18, 28100 Novara, Italy
[b] Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy

Received 17 November 2005; accepted 5 March 2006
Available online 5 May 2006
Submitted by S. Fallat

**Abstract**

It has been empirically observed that correlation matrices of forward interest rates have the first three eigenvalues which are simple and their corresponding eigenvectors, termed as shift, slope and curvature respectively, with elements presenting changes of sign in a regular way. These spectral properties are very similar to those exhibited by Strictly Totally Positive and Oscillatory matrices. In the present paper we investigate how these spectral properties are related with those characterizing the correlation matrices considered, i.e. the positivity and the monotonicity of their elements. On the basis of these relations we prove the simplicity of the first two eigenvalues and provide an estimate of the second one.


*AMS classification:* 15A48; 62H25; 91B28

*Keywords:* Forward rates; Correlation matrices; Principal component analysis; Total positivity

## 1. Introduction

Since the middle of the eighties an intensive effort has been devoted to develop statistical methods in order to describe the movements of the yield curve and to empirically justify some important models based on one or more factors (see e.g. [8,9,20,21]). Principal component analysis (PCA from now on) has turned out to be one of the most useful tools to find such factors.

* Corresponding author. Tel.: +39 02 2399 4546; fax: +39 02 2399 4621.
*E-mail addresses:* ernesto.salinelli@eco.unipmn.it (E. Salinelli), carlo.sgarra@mate.polimi.it (C. Sgarra).

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2006.03.021

Consider a yield curve completely described by an $n$-dimensional vector of forward rates corresponding to maturities $t_1 < t_2 < \cdots < t_n$. These forward rates can be assumed (see [13]) as the components of a random vector (r.v.) $\mathbf{X}^{\mathrm{T}} = [X_1, X_2, \ldots, X_n]$ with zero mean $\mathbb{E}[\mathbf{X}]$ and unit variances $\mathbb{E}[X_i^2]$. This vector can be expressed as a linear transformation $V\mathbf{Y}$ of a $n$-dimensional r.v. $\mathbf{Y}$ with uncorrelated elements (factors) $Y_j = \mathbf{V}_j^{\mathrm{T}}\mathbf{X}$ termed as (see [12]) the *principal components* (PCs) of $\mathbf{X}$. The vectors $\mathbf{V}_j$, for $j = 1, 2, \ldots, n$, are the normalized eigenvectors of the correlation matrix $R = \mathbb{E}[\mathbf{X}'\mathbf{X}]$ of $\mathbf{X}$ and the corresponding eigenvalues $\lambda_j$ are the variances of the PCs, taken in decreasing order: $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. The elements $V_{ij}$ of each vector $\mathbf{V}_j$ represent the contribution of the $i$th original variable to the $j$th PC.

We want to mention here that PCA has been extensively used for applications both to fixed income derivatives pricing and to risk management (as in [15,22]); several indices have been introduced in literature to measure the risk related to yield curve dynamics, and several different approaches have been presented to manage exposure of portfolios to such risk, but many of them have been proved to be related to PCA in some way [7].

An interesting feature of the correlation matrices for forward rates is that they systematically exhibit some structural properties which have a very simple financial explanation:

(a) interest rates at different maturities are always positively correlated;
(b) the correlation coefficients decrease when the distance between the indices increases: this is a far obvious consequence of the decreasing degree of correlation when the variables are more distant in time;
(c) the previous reduction in the correlation between variables corresponding to the same difference in the indices tends to decrease as the maturities of both the variables are greater.

It is important to stress at this point that in the PCA for yield curve models, given the high degree of correlation between the variables involved, a small number of PCs can explain a large part of the variability, i.e. if $n$ is the dimension of $\mathbf{X}$ then

$$\sum_{s=1}^{k} \lambda_s \simeq \operatorname{tr}(R) = n \quad \text{for } k \ll n.$$

More precisely, the first three PCs are usually considered to be sufficient to account for the total variability. Moreover, it has been empirically observed (see [20]) that the first eigenvector $\mathbf{V}_1$ has approximately equal components, hence it is interpreted as the average level of the yield curve and called the **shift** (or **level**); the second eigenvector $\mathbf{V}_2$ has elements approximately equal in magnitude and opposite signs at the extremes of the maturity range and for this reason it is named the **slope** of the yield curve; the third eigenvector $\mathbf{V}_3$ has components approximately equal at the extremes of the maturity range and twice large and of the opposite sign in the middle, hence it is generally called the **curvature**. It has been observed also that the eigenvalues of the correlation matrices appearing in this context are always simple.

In this paper we study how the empirically observed results afore mentioned are related with the structure of the correlation matrix of the data, by using some tools originally developed in the framework of totally positive matrices (TP). TP matrices have been thoroughly investigated in the past: we just mention the books by Gantmacher [3], Gantmacher and Krein [4], Karlin [10] and the more recent review by Ando [1]. In particular, in [4] the applications to continuum mechanics and the relations with oscillating systems have been systematically studied. A more detailed study of the financial implications of the present approach will be the subject of a forthcoming paper.