Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 418 (2006) 775–787
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Matrix units associated with the split basis of a Leonard pair

Kazumasa Nomura [a,*], Paul Terwilliger [b]

[a] *College of Liberal Arts and Sciences, Tokyo Medical and Dental University, Kohnodai, Ichikawa 272-0827, Japan*
[b] *Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, Wisconsin 53706, USA*

Received 21 February 2006; accepted 7 March 2006
Available online 27 April 2006
Submitted by R.A. Brualdi

Abstract

Let $\mathbb{K}$ denote a field, and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i) and (ii) below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

We call such a pair a *Leonard pair* on V. It is known that there exists a basis for V with respect to which the matrix representing A is lower bidiagonal and the matrix representing A^* is upper bidiagonal. In this paper we give some formulae involving the matrix units associated with this basis.

AMS classification: 05E35; 05E30; 33C45; 33D45

Keywords: Leonard pair; Tridiagonal pair; q-Racah polynomial; Orthogonal polynomial

* Corresponding author.
E-mail addresses: nomura.las@tmd.ac.jp (K. Nomura), terwilli@math.wisc.edu (P. Terwilliger).

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2006.03.009

1. Leonard pairs and Leonard systems

We begin by recalling the notion of a Leonard pair. We will use the following terms. A square matrix X is said to be *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume X is tridiagonal. Then X is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. We now define a Leonard pair. For the rest of this paper $\mathbb{K}$ will denote a field.

Definition 1.1 [18]. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair (A, A^*), where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy (i) and (ii) below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Note 1.2. It is a common notational convention to use A^* to represent the conjugate-transpose of A. We are *not* using this convention. In a Leonard pair (A, A^*) the linear transformations A and A^* are arbitrary subject to (i) and (ii) above.

We refer the reader to [3,9,12–18,20–27,29,30] for background on Leonard pairs. We especially recommend the survey [27]. See [1,2,4–8,10,11,19,28] for related topics.

Let A, A^* denote a Leonard pair on V. It is known that A, A^* together generate the $\mathbb{K}$-algebra $\mathrm{End}(V)$ [26, Corollary 6.4]. Therefore each element of $\mathrm{End}(V)$ is a polynomial in A and A^*. However, for any given element of $\mathrm{End}(V)$ it may be unclear what is the corresponding polynomial. In this paper we display a basis Δ_{ij} for the $\mathbb{K}$-vector space $\mathrm{End}(V)$, and express each Δ_{ij} explicitly as a polynomial in A and A^*. The Δ_{ij} are described as follows. By [18, Theorem 3.2] there exists a basis for V with respect to which the matrix representing A is lower bidiagonal and the matrix representing A^* is upper bidiagonal. The elements Δ_{ij} act on this basis as matrix units. The paper also contains some formulae involving the primitive idempotents of A and A^*, which might be of independent interest.

When working with a Leonard pair, it is convenient to consider a closely related object called a *Leonard system*. To prepare for our definition of a Leonard system, we recall a few concepts from linear algebra. Let d denote a nonnegative integer and let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices that have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. For the rest of this paper we let $\mathscr{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$. Let V denote a simple $\mathscr{A}$-module. We remark that V is unique up to isomorphism of $\mathscr{A}$-modules, and that V has dimension $d+1$. Let $v_0, v_1, \ldots, v_d$ denote a basis for V. For $X \in \mathscr{A}$ and $Y \in \mathrm{Mat}_{d+1}(\mathbb{K})$, we say *$Y$ represents X with respect to* $v_0, v_1, \ldots, v_d$ whenever $Xv_j = \sum_{i=0}^{d} Y_{ij} v_i$ for $0 \leqslant j \leqslant d$. For $A \in \mathscr{A}$ we say A is *multiplicity-free* whenever it has $d+1$ mutually distinct eigenvalues in $\mathbb{K}$. Assume A is multiplicity-free. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote an ordering of the eigenvalues of A, and for $0 \leqslant i \leqslant d$ put

$$E_i = \prod_{\substack{0 \leqslant j \leqslant d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j}, \tag{1}$$