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# Tight bounds on the algebraic connectivity of Bethe trees ☆

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**Abstract**

A rooted Bethe tree $\mathscr{B}_{d,k}$ is an unweighted rooted tree of $k$ levels in which the vertex root has degree $d$, the vertices in level 2 to level $(k-1)$ have degree $(d+1)$ and the vertices in level $k$ have degree 1 (pendant vertices). In this paper, we derive tight upper and lower bounds on the algebraic connectivity of

(1) a Bethe tree $\mathscr{B}_{d,k}$, and
(2) a tree $\mathscr{B}_{d,k_1,k_2}$ obtained by the union of two Bethe trees $\mathscr{B}_{d,k_1}$ and $\mathscr{B}_{d,k_2}$ having in common the vertex root.

A useful tool in our study is the Sherman–Morrison formula.


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## 1. Introduction

Let $\mathscr{G}$ be a simple undirected graph on $n$ vertices. The Laplacian matrix of $\mathscr{G}$ is the $n \times n$ matrix $L(\mathscr{G}) = D(\mathscr{G}) - A(\mathscr{G})$ where $A(\mathscr{G})$ is the adjacency matrix of $\mathscr{G}$ and $D(\mathscr{G})$ is the diagonal matrix of vertex degrees. It is well known that $L(\mathscr{G})$ is a positive semidefinite matrix and that $(0, \mathbf{e})$ is an

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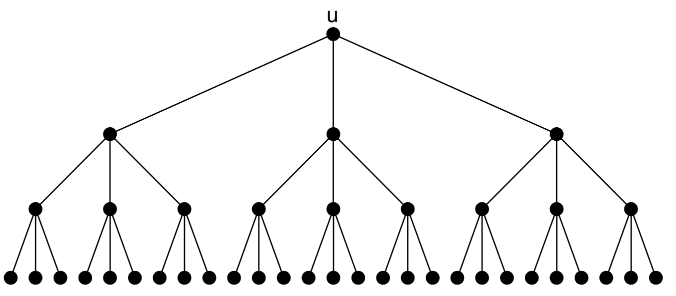
eigenpair of $L(\mathscr{G})$ where $\mathbf{e}$ is the all ones vector. Fiedler [2] proved that $\mathscr{G}$ is a connected graph if and only if the second smallest eigenvalue of $L(\mathscr{G})$ is positive. This eigenvalue is called the algebraic connectivity of $\mathscr{G}$ which is here denoted by $a(\mathscr{G})$.

We recall that a tree is a connected acyclic graph. We recall the notion of a *rooted Bethe tree* $\mathscr{B}_{d,k}$ [4]. The tree $\mathscr{B}_{d,1}$ is a single vertex. For $k > 1$ the tree $\mathscr{B}_{d,k}$ consists of a vertex $u$ which is joined by edges to the roots of each of $d$ copies of $\mathscr{B}_{d,k-1}$. The vertex $u$ is the root of $\mathscr{B}_{d,k}$. We assume $d > 1$.

**Example 1.** The tree $\mathscr{B}_{3,4}$ is




We see that $\mathscr{B}_{3,4}$ is a tree of four levels in which the vertex root has degree equal to 3, the vertices in level 2 and level 3 have degree equal to 4 and the vertices in level 4 have degree equal to 1.

In general, $\mathscr{B}_{d,k}$ is a rooted tree of $k$ levels in which the root vertex has degree equal to $d$, the vertices in level $j$ $(2 \leqslant j \leqslant k-1)$ have degree equal to $(d+1)$ and the vertices in level $k$ (the pendant vertices) have degree equal to 1.

If $d = 2$ then $\mathscr{B}_{2,k}$ is a balanced binary tree of $k$ levels. In [6], Molitierno, Neumann and Shader obtain quite tight upper and lower bounds on the algebraic connectivity of $\mathscr{B}_{2,k}$. The bounds of these authors are

$$a(\mathscr{B}_{2,k}) \leqslant \frac{1}{(2^k - 2k + 3) - \frac{2k-2}{2^{k-1}-1}} \tag{1}$$

and

$$\frac{1}{(2^k - 2k + 2) - \frac{2k-\sqrt{2}(2k-1-2^{k-1})}{2^k-1-\sqrt{2}(2^{k-1}-1)} + \frac{1}{3-2\sqrt{2}\cos\left(\frac{\pi}{2k-1}\right)}} \leqslant a(\mathscr{B}_{2,k}). \tag{2}$$

In this paper, we obtain quite tight upper and lower bounds on the algebraic connectivity of

(1) a tree $\mathscr{B}_{d,k}$, and
(2) a tree $\mathscr{B}_{d,k_1,k_2}$ obtained by the union of two Bethe trees $\mathscr{B}_{d,k_1}$ and $\mathscr{B}_{d,k_2}$ having a common vertex root.

A very useful tool in our study is the Sherman–Morrison formula [1,3] which states that *if $A$ is an $n \times n$ nonsingular matrix and if*

$$B = A + \mathbf{u}\mathbf{v}^{\mathrm{T}},$$

*where $\mathbf{u}$ and $\mathbf{v}$ are $n$-dimensional column vectors, then*