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Sign patterns allowing nilpotence of index 3 ☆

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Abstract

Suppose P is a property referring to a real matrix. We say that a sign pattern A allows P if there exists at least one matrix with the same sign pattern as A that has the property P. In this paper, we study sign patterns allowing nilpotence of index 3. Four methods for constructing sign patterns that allow nilpotence of index 3 are obtained. All tree sign patterns that allow nilpotence of index 3 are characterized. Sign patterns of order 3 that allow nilpotence are identified.

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1. Introduction

The sign of a real number a, denoted by $\mathrm{sgn}(a)$, is defined to be 1, -1 or 0, according to $a > 0$, $a < 0$ or $a = 0$. A *sign pattern matrix* (or a *sign pattern*, for short) is a matrix whose entries are from the set $\{1, -1, 0\}$. The sign pattern of a real matrix B, denoted by $\mathrm{sgn}(B)$, is the sign pattern matrix obtained from B by replacing each entry by its sign.

Let Q_n be the set of all sign patterns of order n. For $A \in Q_n$, the set of all real matrices with the same sign pattern as A is called the *qualitative class* of A, and is denoted by $Q(A)$ [2].

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Suppose P is a property referring to a real matrix. Then a sign pattern A is said to *require* P if every real matrix in $Q(A)$ has property P, or to *allow* P if some real matrix in $Q(A)$ has property P. In this paper, we investigate the property N of being nilpotent. Recall that a real matrix B is said to be *nilpotent* if $B^k = 0$ for some positive integer k. The smallest such integer k is called the *index* (*of nilpotence*) of B.

Let k be a positive integer. We now consider sign patterns that allow $B^k = 0$, that is, we consider sign patterns that allow nilpotence of index at most k. The sign patterns that allow nilpotence, also referred to as the *potentially nilpotent sign patterns* (see [3–5]), form a large class. We denote the class of all sign patterns that allow nilpotence of index at most k by $\mathcal{N}_k$. $\mathcal{N}_2$ is studied in [3]. In this paper, we investigate $\mathcal{N}_3$. In particular, some results in [3] are extended.

Let $A = (a_{ij}) \in Q_n$. A formal nonzero product of the form

$$P = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_{k+1}}$$

is called a *walk* of length k from i_1 to i_{k+1}; if the index set $\{i_1, i_2, \ldots, i_k, i_{k+1}\}$ consists of distinct indices, P is called a *path of length k* (or *k-path*). A formal nonzero product of the form

$$\gamma = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1},$$

in which the index set $\{i_1, i_2, \ldots, i_k\}$ consists of distinct indices is called a *simple cycle of length k* (or *simple k-cycle*). A *composite k-cycle* is a product of simple cycles whose total length is k and whose index sets are mutually disjoint. A cycle (simple or composite) just corresponds to a term in the determinant expansion of the principal submatrix associated with the indices of the cycle. We denote by $c(A)$ the maximum of the lengths of all simple and composite cycles of A.

For $A \in Q_n$, we define the *minimum rank* of A as

$$\mathrm{mr}(A) = \min\{\mathrm{rank}(B) \mid B \in Q(A)\}.$$

A *subpattern* $\hat{A}$ of a sign pattern A is a sign pattern obtained by replacing a number (possibly none) of the nonzero entries in A with 0. We also say that A is a *super-pattern* of $\hat{A}$.

A *permutation pattern* is a square sign pattern with entries 0 and 1, where the entry 1 occurs precisely once in each row and in each column. A *permutational similarity* of the (square) pattern A is a product of the form $S^{\mathrm{T}}AS$, where S is a permutation pattern.

A *signature pattern* is a diagonal sign pattern, each of whose diagonal entries is 1 or −1. A *signature similarity* of the (square) pattern A is a product of the form SAS, where S is a signature pattern.

2. Some basic results

Lemma 2.1 [3]. *The set $\mathcal{N}_k$ is closed under the following operations*:

(i) *negation*,
(ii) *transposition*,
(iii) *permutational similarity*, *and*
(iv) *signature similarity*.

In this paper, we say two sign patterns are *equivalent* if one can be obtained from the other by performing a sequence of operations listed in Lemma 2.1. This is indeed an equivalence relation.

Lemma 2.2. *A real matrix B is nilpotent if and only if each of its eigenvalues is equal to zero.*