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An asymptotic behavior of *QR* decomposition

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Abstract

The m-th root of the diagonal of the upper triangular matrix R_m in the *QR* decomposition of $AX^mB = Q_mR_m$ converges and the limit is given by the moduli of the eigenvalues of X with some ordering, where $A, B, X \in \mathbb{C}_{n\times n}$ are nonsingular. The asymptotic behavior of the strictly upper triangular part of R_m is discussed. Some computational experiments are discussed.

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1. Introduction

The *QR* decomposition [4] of a nonsingular $X \in \mathbb{C}_{n\times n}$ asserts that

$$X = QR,$$

where $Q \in \mathbb{C}_{n\times n}$ is unitary and $R \in \mathbb{C}_{n\times n}$ is upper triangular with positive diagonal entries and the decomposition is unique. It is simply a matrix version of the traditional Gram–Schmidt process on the columns of X. The diagonal entries of R have very nice geometric interpretation, that is, r_{ii}, $i = 1, \ldots, n$, is equal to the distance from the i-th column of X to the space spanned by the first $i-1$ columns of X. We denote by

$$a(X) := \mathrm{diag}(a_1(X), \ldots, a_n(X)) = \mathrm{diag}(r_{11}, \ldots, r_{nn}), \quad (1)$$

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the diagonal matrix of R. In this paper, it is shown in Section 2 that given nonsingular $A, B, X \in \mathbb{C}_{n\times n}$, and the QR decomposition $AX^mB = Q_mR_m$, the sequence of matrices

$$\{[a(AX^mB)]^{1/m}\}_{m=1}^{\infty} = \{(\operatorname{diag} R_m)^{1/m}\}_{m=1}^{\infty} \tag{2}$$

converges and the limit is given by the moduli of the eigenvalues of X. The asymptotic behavior of the strictly upper triangular part of R_m is studied in Section 3. Some computational experiments using MAPLE and MATLAB are discussed in the last section.

2. Convergence of $[a(AX^mB)]^{1/m}$

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{C}^n$, that is, $\mathbf{e}_i$ has 1 as the only nonzero entry at the i-th position. We identify a permutation $\omega \in S_n$ with the unique permutation matrix (also written as ω) in the general linear group $GL_n(\mathbb{C})$, where $\omega\mathbf{e}_i = \mathbf{e}_{\omega(i)}$. The matrix representation of ω under the standard basis is

$$\omega = [\mathbf{e}_{\omega(1)}, \ldots, \mathbf{e}_{\omega(n)}].$$

Given a matrix $A \in \mathbb{C}_{n\times n}$, let $A(i|j)$ denote the submatrix formed by the first i rows and the first j columns of A, $1 \leqslant i, j \leqslant n$.

Theorem 2.1. *Let $A, B, X \in GL_n(\mathbb{C})$. Let $X = Y^{-1}DY$ be the Jordan decomposition of X, where D is the Jordan form of X, $\operatorname{diag} D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ satisfying $|\lambda_1| \geqslant \cdots \geqslant |\lambda_n|$.*
Then

$$\lim_{m\to\infty} a(AX^mB)^{1/m} = \operatorname{diag}(|\lambda_{\omega(1)}|, \ldots, |\lambda_{\omega(n)}|), \tag{3}$$

where the permutation ω is uniquely determined by YB:

$$\operatorname{rank} \omega(i|j) = \operatorname{rank}(YB)(i|j) \quad \text{for } 1 \leqslant i, j \leqslant n. \tag{4}$$

Proof. Let $X_m := AX^mB$ have the QR decomposition $X_m = Q_mR_m$. Then

$$a_1(X_m)\cdots a_k(X_m) = \det R_m(k|k) = \sqrt{\det(X_m(n|k)^*X_m(n|k))}. \tag{5}$$

That is, the product $a_1(X_m)\cdots a_k(X_m)$ is uniquely determined by the first k columns of X_m.

Set $D_0 := \operatorname{diag} D$. Then $D = CD_0$ for a unit upper triangular matrix C commuting with D_0. By LU decomposition [2, p. 164], $YB = L\omega U$, for some (unique) permutation matrix ω, some unit lower triangular matrix L, and some nonsingular upper triangular matrix U. By block multiplication

$$\begin{aligned}(YB)(i|j) &= \begin{bmatrix} L(i|i) & 0 \end{bmatrix} \begin{bmatrix} \omega(i|j) & * \\ * & * \end{bmatrix} \begin{bmatrix} U(j|j) \\ 0 \end{bmatrix} \\ &= L(i|i)\omega(i|j)U(j|j).\end{aligned}$$

So ω satisfies $\operatorname{rank} \omega(i|j) = \operatorname{rank}(YB)(i|j)$ for $1 \leqslant i, j \leqslant n$. Obviously $\operatorname{rank} \omega(i|j)$ is the number of nonzero entries in $\omega(i|j)$. Thus it is easy to verify that ω_{ij} is a nonzero entry 1 if and only if

$$\operatorname{rank} \omega(i|j) - \operatorname{rank} \omega(i|j-1) - \operatorname{rank} \omega(i-1|j) + \operatorname{rank} \omega(i-1|j-1) = 1.$$

So the permutation matrix ω is uniquely determined by $\operatorname{rank} \omega(i|j)$, $1 \leqslant i, j \leqslant n$. Hence ω is uniquely determined by YB.