Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 422 (2007) 295–303
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Chebyshev type inequalities involving permanents and their applications ☆

Jiajin Wen *, Wan-lan Wang

Department of Mathematics and Computer Science, Chengdu University, Chengdu, Sichuan 610106, PR China

Received 17 October 2005; accepted 17 October 2006
Available online 6 December 2006
Submitted by S. Friedland

Abstract

We extend the well-known Chebyshev's inequality to some new cases involving permanents under the proper hypotheses. Our main results are

$$\frac{\operatorname{per}(A \odot B)}{n!} \geqslant \frac{\operatorname{per} A}{n!} \cdot \frac{\operatorname{per} B}{n!} \quad \text{and} \quad \frac{\operatorname{per} A}{\prod_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}} \leqslant \frac{\operatorname{per} B}{\prod_{i=1}^{n} \sum_{j=1}^{n} b_{i,j}}.$$

As applications, we consider the problem of finding bounds of the ratios of two functions involving permanents.

AMS classification: 15D45; 15A15

Keywords: Chebyshev type inequality; Permanent; Generalization; Bound

☆ Supported by the NSF of China (No. 10171073).
* Corresponding author.
E-mail addresses: jiajinwen623@163.com (J.J. Wen), wanlanwang@163.com (W.-l. Wang).

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2006.10.014

1. Introduction and main results

We shall need the following symbols in the well-known monographs [1–3]:

$$x := (x_1, x_2, \ldots, x_n); \quad \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n);$$
$$xy := (x_1y_1, x_2y_2, \ldots, x_ny_n);$$
$$x/y := (x_1/y_1, x_2/y_2, \ldots, x_n/y_n), \quad y_i \neq 0 \ \ (i = 1, \ldots, n);$$
$$\mathbb{R}^n :=]-\infty, \infty[^n; \quad \mathbb{R}^n_+ := [0, \infty[^n; \quad \mathbb{R}^n_{++} :=]0, \infty[^n.$$

For matrices $A = (a_{i,j})_{m\times n}$ and $B = (b_{i,j})_{m\times n}$, we define the Haddamard product as $A \odot B := (a_{i,j}b_{i,j})_{m\times n}$, that is, it is the componentwise product.

Besides these, throughout the paper it is assumed that $n \geqslant 2$.

The well-known Chebyshev's inequality states: if $a, b \in \mathbb{R}^n$, and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$, then

$$\frac{1}{n}\sum_{i=1}^n a_i b_i \geqslant \left(\frac{1}{n}\sum_{i=1}^n a_i\right)\left(\frac{1}{n}\sum_{i=1}^n b_i\right). \tag{1}$$

The inequality is reversed for $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$. In each case, equality holds in (1) if and only if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

Definition 1. Let $A = (a_{i,j})_{n\times n}$ be an $n \times n$ matrix over a commutative ring. Then the permanent (of order n) of A, written per$A$, is defined by

$$\operatorname{per}A := \sum_{\sigma\in S_n} a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}.$$

The sum here extends over all elements σ of the symmetric group S_n, that is, over all permutations of the numbers $1, 2, \ldots, n$.

We shall use some symbols similar to those in [2,3]: let $A = (a_{i,j})_{m\times n}$ be an $m \times n$ matrix. Then $A(i_1, i_2, \ldots, i_p|j_1, j_2, \ldots, j_q)$ denotes the $(m - p) \times (n - q)$ submatrix obtained from A by deleting its i_1th, i_2th, $\ldots$, i_pth rows and j_1th, j_2th, $\ldots$, j_qth columns. For any n-square matrix $A = (a_{i,j})_{n\times n}$, we will use the following lemma similar to Laplace's expansion theorem for determinants (see [2,3]):

Lemma 1. *The expansion of the permanent according to the rth row or the sth column*

$$\operatorname{per}A = \sum_{j=1}^n a_{r,j}\operatorname{per}A(r|j) = \sum_{i=1}^n a_{i,s}\operatorname{per}A(i|s) \quad (r, s = 1, 2, \ldots, n).$$

We shall generalize the inequality (1) to the following results (4) and (6).

Theorem 1. *Let* $A = (a_{i,j})_{n\times n}$ *and* $B = (b_{i,j})_{n\times n}$ *be two* $n \times n$ *matrices, and let* $a_{i,j} > 0$ *and* $b_{i,j} > 0$, $i, j = 1, 2, \ldots, n$. *If*

$$\frac{a_{i,1}}{a_{i+1,1}} \leqslant \frac{a_{i,2}}{a_{i+1,2}} \leqslant \cdots \leqslant \frac{a_{i,n}}{a_{i+1,n}}, \quad i = 1, 2, \ldots, n-1, \tag{2}$$