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The spectra of a graph obtained from copies of a generalized Bethe tree ☆

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Abstract

We generalize the concept of a Bethe tree as follows: we say that an unweighted rooted tree is a generalized Bethe tree if in each level the vertices have equal degree. If $\mathscr{B}_k$ is a generalized Bethe tree of k levels then we characterize completely the eigenvalues of the adjacency matrix and Laplacian matrix of a graph $\mathscr{B}_k^{(r)}$ obtained from the union of r copies of $\mathscr{B}_k$ and the cycle $\mathscr{C}_r$ connecting the r vertex roots. Moreover, we give results on the multiplicity of the eigenvalues, on the spectral radii and on the algebraic conectivity.

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Keywords: Tree; Bethe tree; Laplacian matrix; Adjacency matrix; Algebraic connectivity

1. Preliminaries

Let $\mathscr{G}$ be a simple undirected graph on n vertices. The Laplacian matrix of $\mathscr{G}$ is the $n \times n$ matrix $L(\mathscr{G}) = D(\mathscr{G}) - A(\mathscr{G})$ where $A(\mathscr{G})$ is the adjacency and $D(\mathscr{G})$ is the diagonal matrix of vertex degrees. It is well known that $L(\mathscr{G})$ is a positive semidefinite matrix and that $(0, \mathbf{e})$ is an eigenpair of $L(\mathscr{G})$ where $\mathbf{e}$ is the all ones vector. Fiedler [4] proved that $\mathscr{G}$ is a connected graph if and only if the second smallest eigenvalue of $L(\mathscr{G})$ is positive. This eigenvalue is called the algebraic connectivity of $\mathscr{G}$.

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A tree is a connected acyclic graph.

A *Bethe tree* [2] is an unweighted rooted tree of k levels such that the vertex root has degree d, the vertices in the intermediate levels have degree $(d+1)$ and the vertices in level k are the pendant vertices.

We generalize the notion of a Bethe tree as follows: we say that an unweighted rooted tree $\mathscr{B}_k$ of k levels is a *generalized Bethe tree* if in each level the vertices have equal degree.

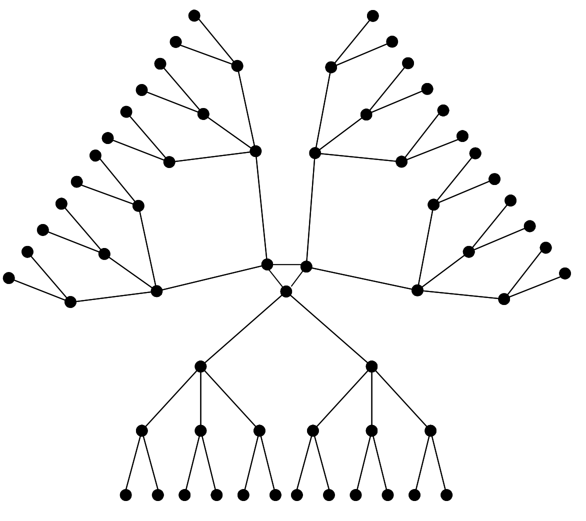
Throughout this paper $\mathscr{B}_k$ denotes a generalized Bethe tree of k levels with $k > 1$.

In [7, 2005], we found the eigenvalues of the adjacency matrix and Laplacian matrix of $\mathscr{B}_k$. They are the eigenvalues of leading principal submatrices of two nonnegative symmetric tridiagonal matrices of order $k \times k$ whose entries are given in terms of the vertex degrees.

In [8, 2006], we characterized completely the eigenvalues of the adjacency matrix and Laplacian matrix of a tree $\mathscr{B}_k^{(2)}$ obtained from the union of two copies of $\mathscr{B}_k$ and the edge connecting the two vertex roots. If we agree that the two vertex roots are at level 1 and d_{k-j+1} denotes the degree of the vertices in level j, then they are the eigenvalues of leading principal submatrices of nonnegative symmetric tridiagonal matrices of order $k \times k$. The codiagonal entries for these matrices are $\sqrt{d_j - 1}$, $2 \leqslant j \leqslant k$, while the diagonal entries are $0, \ldots, 0, \pm 1$, in the case of the adjacency matrix, and $d_1, d_2, \ldots, d_{k-1}, d_k \pm 1$, in the case of the Laplacian matrix.

Let r be a positive integer greater or equal to 3.

In this paper, we search for the eigenvalues of the adjacency matrix and Laplacian matrix of a graph $\mathscr{B}_k^{(r)}$ obtained from the union of r copies of $\mathscr{B}_k$ and the cycle $\mathscr{C}_r$ connecting the r vertex roots. An example of a such graph is



In this graph, we observe four levels of vertices:

3 vertices in the cycle $\mathscr{C}_3$, say in level 1, each of them with degree equal to 4,

6 vertices in level 2, each of them with degree equal to 4,

18 vertices in level 3, each of them with degree equal to 3, and

36 pendant vertices in level 4.

In general, $\mathscr{B}_k^{(r)}$ is a graph of k levels such that in each level the vertices have equal degree. We agree that the vertices of $\mathscr{C}_r$ are at level 1. For $j = 1, 2, \ldots, k$, let d_{k-j+1} and n_{k-j+1} be the degree of the vertices and number of them in level j. Thus, d_k is the degree of the vertices in