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Linear Algebra and its Applications 420 (2007) 625–633
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

A nontrivial upper bound on the largest Laplacian eigenvalue of weighted graphs ☆

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Received 11 May 2006; accepted 24 August 2006
Available online 10 October 2006
Submitted by R.A. Brualdi

Abstract

Let $\mathscr{G}$ be a simple connected weighted graph on n vertices in which the edge weights are positive numbers. Denote by $i \sim j$ if the vertices i and j are adjacent and by $w_{i,j}$ the weight of the edge ij. Let $w_i = \sum_{j=1}^{n} w_{i,j}$. Let λ_1 be the largest Laplacian eigenvalue of $\mathscr{G}$. We first derive the upper bound

$$\lambda_1 \leqslant \sum_{j=1}^{n} \max_{k \sim j} w_{k,j}.$$

We call this bound the trivial upper bound for λ_1. Our main result is

$$\lambda_1 \leqslant \frac{1}{2} \max_{i \sim j} \left\{ \begin{array}{c} w_i + w_j + \sum_{k \sim i, k \nsim j} w_{i,k} + \sum_{k \sim j, k \nsim i} w_{j,k} \\ + \sum_{k \sim i, k \sim j} |w_{i,k} - w_{j,k}| \end{array} \right\}.$$

For any $\mathscr{G}$, this new bound does not exceed the trivial upper bound for λ_1.

AMS classification: 05C50

Keywords: Graph; Weighted graph; Laplacian matrix; Spectral radius; Upper bound

☆ Work supported by Fondecyt 1040218, Chile.
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doi:10.1016/j.laa.2006.08.022

1. Preliminaries

We consider a simple connected weighted graph in which the edge weights are positive numbers. Let $\mathscr{G}$ be such a graph. Labelling the vertices of $\mathscr{G}$ by $1, 2, \ldots, n$, denote by $w_{i,j}$ the weight of the edge ij. We write $i \sim j$ if the vertices i and j are adjacent. Let $w_i = \sum_{k \sim i} w_{k,i}$. The Laplacian matrix of $\mathscr{G}$ is the $n \times n$ matrix $L(\mathscr{G}) = (l_{i,j})$ defined by

$$l_{i,j} = \begin{cases} w_i & \text{if } i = j, \\ -w_{i,j} & \text{if } i \sim j, \\ 0 & \text{if } i \nsim j,\ i \neq j. \end{cases}$$

$L(\mathscr{G})$ is a real symmetric matrix. From this fact and Geršgorin's Theorem, it follows that the eigenvalues of $L(\mathscr{G})$ are nonnegative real numbers. Since each row sum is 0, $(0, \mathbf{e})$ is an eigenpair for $L(\mathscr{G})$ where $\mathbf{e}$ is the all ones vector. Moreover, $\mathscr{G}$ is a connected graph if and only if 0 is a simple eigenvalue. We assume that $\mathscr{G}$ is a connected graph.

Throughout this paper we assume that

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} > \lambda_n = 0$$

are the eigenvalues of $L(\mathscr{G})$.

If $w_{i,j} = 1$ for all edge ij then $\mathscr{G}$ is an unweighted graph. In [1], some of the many results known for the Laplacian matrix of an unweighted graph are given. Upper bounds for λ_1 in the case of unweighted graphs have been obtained by several authors [2–10].

We recall the following result due to Brauer [11]:

Theorem 1. *Let A be an $n \times n$ arbitrary matrix with eigenvalues*

$$\lambda_1, \lambda_2, \ldots, \lambda_n.$$

Let

$$\mathbf{v} = [v_1, v_2, \ldots, v_n]^{\mathrm{T}}$$

be an eigenvector of A corresponding to the eigenvalue λ_k and let $\mathbf{q}$ be any n-dimensional column vector. Then the matrix $A + \mathbf{v}\mathbf{q}^{\mathrm{T}}$ has eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k + \mathbf{v}^T\mathbf{q}, \lambda_{k+1}, \ldots\ldots, \lambda_n.$$

We define the vector

$$\mathbf{p} = [p_1, p_2, \ldots, p_n]^{\mathrm{T}}$$

where, for $j = 1, 2, \ldots, n$,

$$p_j = \max_{k \sim j} w_{k,j}.$$

Then $p_j - w_{i,j} \geqslant 0$ for all $i \sim j$. We now define the matrix

$$B(\mathscr{G}) = L(\mathscr{G}) + \mathbf{e}\mathbf{p}^{\mathrm{T}}. \tag{1}$$

The entries of $B(\mathscr{G})$ are

$$b_{i,j}(\mathscr{G}) = \begin{cases} w_i + p_i & \text{if } i = j, \\ -w_{i,j} + p_j & \text{if } i \sim j, \\ p_j & \text{if } i \nsim j, i \neq j. \end{cases} \tag{2}$$