



L_p Linear discrepancy of totally unimodular matrices

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Received 24 June 2003; accepted 28 August 2006

Available online 9 October 2006

Submitted by R.A. Brualdi

Abstract

Let $p \in [1, \infty[$ and $c_p = \max_{a \in [0,1]} ((1-a)a^p + a(1-a)^p)^{1/p}$. We prove that the known upper bound $\text{lindisc}_p(A) \leq c_p$ for the L_p linear discrepancy of a totally unimodular matrix A is asymptotically sharp, i.e.,

$$\sup_A \text{lindisc}_p(A) = c_p.$$

We estimate $c_p = \frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p} (1 + \varepsilon_p)$ for some $\varepsilon_p \in [0, 2^{-p+2}]$, hence $c_p = 1 - \frac{\ln p}{p} (1 + o(1))$. We also show that an improvement for smaller matrices as in the case of L_∞ linear discrepancy cannot be expected. For any $p \in \mathbb{N}$ we give a totally unimodular $(p+1) \times p$ matrix having L_p linear discrepancy greater than $\frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p}$.

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AMS classification: 11K38

Keywords: Linear discrepancy; Totally unimodular matrix; Rounding; Halftoning

1. Introduction and results

Let $p \in [1, \infty[$ and let $A \in \mathbb{R}^{m \times n}$. Denote the rows of A by $a^{(1)}, \dots, a^{(m)} \in \mathbb{R}^n$. Let $x \in [0, 1]^n$. The L_p linear discrepancy of A with respect to x is

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$$\text{lindisc}_p(A, x) = \min_{y \in \{0,1\}^n} \frac{1}{m^{1/p}} \|A(x - y)\|_p = \min_{y \in \{0,1\}^n} \left(\frac{1}{m} \sum_{i=1}^m |a^{(i)} \cdot (x - y)|^p \right)^{1/p}.$$

The L_p linear discrepancy of A is $\text{lindisc}_p(A) = \max_{x \in [0,1]^n} \text{lindisc}_p(A, x)$.

The matrix A is called *totally unimodular*, if each square submatrix has determinant -1 , 0 or 1 . In particular, the entries of a totally unimodular matrix are from $\{-1, 0, 1\}$. Put $c_p = \max_{a \in [0,1]} ((1-a)a^p + a(1-a)^p)^{1/p}$. Motivated by an application in image processing, Asano et al. [2] (cf. also the survey Asano [1]) show and estimate

$$\text{lindisc}_p(A) \leq c_p \leq 1 - \frac{1}{p+1}. \quad (1)$$

They also note that the n -dimensional identity matrix I_n satisfies $\text{lindisc}_p(I_n) = \frac{1}{2}$ for all p and n . This shows that the first inequality in (1) is sharp for $p \leq 3$.

The objective of this note is to improve the lower bound for $p \geq 3$. We show that for all $n \in \mathbb{N}$, there is a totally unimodular matrix $A \in \{0, 1\}^{(n+1) \times n}$ such that

$$\text{lindisc}_p(A) \geq c_p(1 + o(1))$$

with $o(1)$ term depending on n only. Estimating c_p tighter than in [2] yields

$$\frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p} \leq c_p \leq \frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p} (1 + 2^{-p+2}).$$

Thus $c_p = 1 - \frac{\ln p}{p}(1 + o(1))$. Finally, we give for any $p \in \mathbb{N}$ a totally unimodular $(p+1) \times p$ matrix A such that $\text{lindisc}_p(A) > \frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p}$. This shows that an improvement of (1) for smaller matrices as was recently proven for the L_∞ linear discrepancy (see Section 3) cannot exist.

2. Proofs of the main results

Note first that $c_p = \max_{a \in [0, \frac{1}{2}]} ((1-a)a^p + a(1-a)^p)^{1/p}$ due to symmetry. Since the case $p \leq 3$ was already completely solved in [2], we assume $p \geq 3$ in the following. We prove the following lower bound.

Theorem 1. *For all $p \geq 3$ and all $n \in \mathbb{N}$, there is a totally unimodular matrix $A \in \{0, 1\}^{(n+1) \times n}$ such that*

$$\text{lindisc}_p(A) \geq c_p(1 + o(1))$$

with $o(1)$ term depending on n only.

Proof. Let $a \in]0, \frac{1}{2}]$ and $n \in \mathbb{N}$ be sufficiently large. Define $A \in \{0, 1\}^{(n+1) \times n}$ by $a_{ij} = 1$ if and only if $i = j$ or $i = n+1$. Clearly, A is totally unimodular. Let $x = a\mathbf{1}_n \in \mathbb{R}^n$. Let $y \in \{0, 1\}^n$ such that $\|A(x - y)\|_p$ is minimal. Let k be the number of $i \in \{1, \dots, n\}$ such that $y_i = 1$. Then

$$\|A(x - y)\|_p^p = (n - k)a^p + k(1 - a)^p + |na - k|^p =: f(k).$$

Note that this value only depends on the number k , but not on the distribution of the ones in y . f viewed as function on the reals is convex and has a minimum at $k_0 = an - \left(\frac{(1-a)^p - a^p}{p} \right)^{1/(p-1)}$. This yields

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