Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER Linear Algebra and its Applications 412 (2006) 161–221

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Trace-minimal graphs and D-optimal weighing designs

Bernardo M. Ábrego, Silvia Fernández-Merchant, Michael G. Neubauer, William Watkins *

Department of Mathematics, California State University—Northridge, Northridge, CA 91330-8313, United States

Received 10 May 2004; accepted 20 June 2005
Available online 24 August 2005
Submitted by R.A. Brualdi

Abstract

Let $\mathscr{G}(v, \delta)$ be the set of all δ-regular graphs on v vertices. Certain graphs from among those in $\mathscr{G}(v, \delta)$ with maximum girth have a special property called trace-minimality. In particular, all strongly regular graphs with no triangles and some cages are trace-minimal. These graphs play an important role in the statistical theory of D-optimal weighing designs.

Each weighing design can be associated with a (0, 1)-matrix. Let $M_{m,n}(0, 1)$ denote the set of all $m \times n$ (0, 1)-matrices and let

$$G(m, n) = \max\left\{\det X^{\mathrm{T}}X : X \in M_{m,n}(0, 1)\right\}.$$

A matrix $X \in M_{m,n}(0, 1)$ is a D-optimal design matrix if $\det X^{\mathrm{T}}X = G(m, n)$. In this paper we exhibit some new formulas for $G(m, n)$ where $n \equiv -1 \pmod 4$ and m is sufficiently large. These formulas depend on the congruence class of $n \pmod n$. More precisely, let $m = nt + r$ where $0 \leqslant r < n$. For each pair n, r, there is a polynomial $P(n, r, t)$ of degree n in t, which depends only on n, r, such that $G(nt + r, n) = P(n, r, t)$ for all sufficiently large t. The polynomial $P(n, r, t)$ is computed from the characteristic polynomial of the adjacency matrix of a trace-regular graph whose degree of regularity and number of vertices depend only on n and r. We obtain explicit expressions for the polynomial $P(n, r, t)$ for many pairs n, r. In particular we obtain formulas for $G(nt + r, n)$ for $n = 19$, 23, and 27, all $0 \leqslant r < n$, and all

* Corresponding author.
E-mail address: bill.watkins@csun.edu (W. Watkins).

doi:10.1016/j.laa.2005.06.027

sufficiently large t. And we obtain families of formulas for $P(n, r, t)$ from families of trace-minimal graphs including bipartite graphs obtained from finite projective planes, generalized quadrilaterals, and generalized hexagons.

AMS classification: Primary: 05C50, 62K05; Secondary: 15A36, 05B25, 15A15, 05B20

Keywords: D-optimal weighing design; Trace-minimal graph; Regular graph; Strongly regular graph; Girth; Cages; Generalized polygons

1. Introduction

In [1], the present authors established a relationship between certain regular graphs and D-optimal designs for weighing $n \equiv -1 \pmod 4$ objects. We now further develop the graph-theoretic concept of trace-minimality and use it to obtain additional results on D-optimality.

1.1. Trace-minimal graphs

Let $\mathscr{G}(v, \delta)$ be the set of all δ-regular graphs on v vertices. We call $\mathscr{G}(v, \delta)$ a *graph class*. Let $A(G)$ be the adjacency matrix of a graph G. The characteristic polynomial of $A(G)$ is denoted by $\mathrm{ch}(G, x)$ and the spectrum of $A(G)$ is denoted by $\mathrm{spec}(A(G))$. We also refer to $\mathrm{ch}(G, x)$ as the characteristic polynomial of the graph G and $\mathrm{spec}(A(G))$ as the spectrum of G. Since $A(G)$ is a symmetric (0, 1)-matrix with zeros on the diagonal and δ ones in each row, $\mathrm{tr}\, A(G) = 0$ and $\mathrm{tr}\, A(G)^2 = \delta v$. These traces do not depend on the structure of the graph G. However, for $i \geqslant 3$, $\mathrm{tr}\, A(G)^i$ does depend on the structure of the graph. Indeed the (j, j) entry of $A(G)^i$ equals the number of closed walks of length i that start and end at vertex j. For $G \in \mathscr{G}(v, \delta)$ define the *trace sequence* of G by $\mathrm{TR}(G) = \left(\mathrm{tr}\, A(G)^3, \mathrm{tr}\, A(G)^4, \ldots, \mathrm{tr}\, A(G)^n\right)$.

The trace sequence induces an order relation on the graphs in $\mathscr{G}(v, \delta)$. Let $G, H \in \mathscr{G}(v, \delta)$. We say G is *trace-dominated* by H if $\mathrm{TR}(G)$ is less than or equal to $\mathrm{TR}(H)$ in lexicographic order. In other words, G is trace-dominated by H if either $\mathrm{tr}\, A(G)^i = \mathrm{tr}\, A(H)^i$ for all i (in which case $\mathrm{spec}(A(G)) = \mathrm{spec}(A(H))$) or there exists a positive integer $3 \leqslant k \leqslant n$ such that $\mathrm{tr}\, A(G)^i = \mathrm{tr}\, A(H)^i$, for $i < k$ and $\mathrm{tr}\, A(G)^k < \mathrm{tr}\, A(H)^k$. If G is trace-dominated by all graphs in $\mathscr{G}(v, \delta)$, then we say that G is *trace-minimal* in $\mathscr{G}(v, \delta)$. Since $\mathscr{G}(v, \delta)$ is finite, there always exist trace-minimal graphs in $\mathscr{G}(\delta, v)$ and clearly they all have the same characteristic polynomial. Some graph classes $\mathscr{G}(v, \delta)$, contain non-isomorphic graphs, each of which is trace-minimal. However, the smallest example known to us are two non-isomorphic cages in the graph class $\mathscr{G}(70, 3)$. (See Sections 4.5 and 4.4.1.)

In addition to their application to the theory of D-optimal weighing designs, trace-minimal graphs are of independent interest. Indeed many well-known classes of