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Optimal regularity for phase transition problems with convection

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Abstract

In this paper we consider a steady state phase transition problem with given convection $\mathbf{v}$. We prove, among other things, that the weak solution is locally Lipschitz continuous provided that $\mathbf{v} = D\xi$ and ξ is a harmonic function. Moreover, for continuous casting problem, i.e. when $\mathbf{v}$ is constant vector, we show that Lipschitz free boundaries are C^1 regular surfaces.

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1. Introduction

In this paper we study a stationary phase transition problem where the liquid phase is in motion. For given convection $\mathbf{v}$, the problem is of determining the temperature T from the equation

$$\Delta T = \operatorname{div}\big[\mathbf{v}\beta(T - T_S)\big] + f. \tag{1.1}$$

Here $\beta(s) = as + \ell H(s)$ is the enthalpy, a is the specific heat constant, ℓ is the latent heat constant, H is the Heaviside function, T_S is the solidification temperature and f is a given function that accounts for heat sources or sinks. As one can see, (1.1) is the heat balance equation written for the enthalpy β [21]. (1.1) is also known as Stefan problem with convection.

It is well known that (1.1) portrays various phase transition models. For instance, if $\mathbf{v}$ is constant then we have the so-called *continuous casting problem*, which is a practical example of a free boundary problem appearing in industry [1,6,13,22]. It models a metal fabrication technique whereby molten metal is poured into an open mold and subsequently cooled by a stream of water and extracted at continuous velocity. This method is used most frequently to cast steel, aluminum and copper, because it allows low cost production of metal sections of good quality [25]. Another

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example of this sort is phase change in saturated porous media. For more details concerning the physical background of this equation, see [1,6] and references therein.

If we suppose that $T_S = 0$ then (1.1) transforms into

$$\Delta T = \mathrm{div}\big[\mathbf{v}\beta(T)\big] + f.$$

Notice that β has abrupt behavior at $s = 0$. Typically $\beta(T) = \ell + \int_{T_S}^{T} a(\tau)\,d\tau$ for $T > T_S$ and $\beta(T) = \int_{T_S}^{T} a(\tau)\,d\tau$ if $T < T_S$. Here a is the specific heat which in this paper is assumed to be constant. Hence the solid region is characterized by $T < T_S$ and the liquid region by $T > T_S$. Any region where $T = T_S$ and $0 < \beta(T) < \ell$ is called mushy region [1,19]. The presence of mushy region means that we do not have sharp separation of phases. There are several boundary conditions guaranteeing that the mushy region is empty, see [13,19]. However, our primary interest here is the boundary of the sets $T > T_S$ or $T < T_S$. To fix the ideas we consider $\Gamma = \partial\{T < T_S\}$ which we call the *free boundary* and study its properties. Our methods can be equally applied to the set $\partial\{T > T_S\}$.

The objective of this paper is to prove that weak solutions of (1.1) are locally Lipschitz continuous. Moreover the Lipschitz free boundary must be C^1 smooth, see Theorems A, B and C below.

The phase transition problem with convections has been studied by several authors, see [21] and references therein. The existence of $W^{1,2}$ weak solutions to various boundary value problems for (1.1) can be established by penalization method [13,16,21]. In this way one obtains a bounded Hölder continuous solution for a suitable boundary data. Our first result, Theorem A, strengthens this result up to log-Lipschitz continuity under some weak conditions on the boundary of the domain and the boundary data. However the optimal regularity of the solutions is Lipschitz as the free boundary condition (7.5) indicates. One of our main results in this paper is the local Lipschitz continuity of weak solutions for one phase and two phase problems, see Theorem B. It should be noted that Theorem B does not follow from Theorem A in [5], since we do not assume that the free boundary is given by the graph of a Lipschitz continuous function.

Having proven the optimal regularity of weak solution, we address the free boundary regularity which is a very delicate problem. To tackle it, we apply the free boundary regularity theory for viscosity solutions. The latter is yet another notion of generalized solution, which utilizes the maximum principle at the regular (in some weak sense) free boundary points via a Hopf type lemma. This method was developed by L. Caffarelli for the pure Laplace operator in the series of papers [7,8]. Extension to more general class of operators is proven by M. Feldman in [15]. In view of these results the regularity problem reduces to the equivalence of weak solution to the viscosity solution which is contained in Theorem C.

2. Outline

The paper is organized as follows. In Section 3 we introduce some notations used throughout the paper. In Section 4 we state the phase change problem with convection and give its weak formulation. It is worthwhile to point out that one phase problem is linked to obstacle problem as the computation (4.4) shows. In particular we get that the positive part of weak solutions to continuous casting problem, i.e. when $\mathbf{v} = \mathbf{e}_N$, are locally non-degenerate, see Proposition 2.

The main results of this paper, Theorems A, B and C, are formulated in Section 5. First we show that the weak solution is locally log-Lipschitz continuous. This improves the known result that $u = T_S - T$ is α-Hölder continuous for any $\alpha < 1$. Under further assumption that the lateral boundary $\Sigma = \partial\Omega \times (0, L)$ is Liapunov–Dini surface and the Dirichlet data prescribed on Σ is $C^{0,1}$ we show that the log-Lipschitz estimate holds in $\overline{\mathcal{C}}_L = \overline{\Omega \times (0, L)}$. As a result one obtains that the free boundary is a log-Lipschitz graph over $\overline{\Omega} \subset \mathbb{R}^{N-1}$. This is contained in Theorem A and the proof is given in Section 6.

If we have sharp separation of solid and liquid phases, i.e. the interface does not have thickness, then one can deduce the free boundary condition for smooth solutions directly from Eq. (4.5). This is carried out in Section 7.

In Section 8, we prove that the weak solutions to one phase problem are Lipschitz continuous on any subdomain of $\mathcal{C}_L = \Omega \times (0, L)$. According to the free boundary condition (7.5) the Lipschitz regularity of free boundary is optimal. The rest of Section 8 deals with the one phase continuous casting problem, i.e. when $u \geqslant 0$. Using a strong connection with the obstacle problem we show that u is non-degenerate at free boundary points. This implies that the free boundary $\partial\{u > 0\}$ is locally a set of finite perimeter. Moreover, it is $N - 1$ rectifiable. In Section 9 by employing Alt–Caffarelli–Friedman monotonicity theorem we prove optimal regularity for the solutions of the two phase problem. Note that the proofs of Lipschitz continuity for one and two phase problems differ considerably.