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Regularity for degenerate two-phase free boundary problems

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Abstract

We provide a rather complete description of the sharp regularity theory to a family of heterogeneous, two-phase free boundary problems, $\mathcal{J}_\gamma \to \min$, ruled by nonlinear, p-degenerate elliptic operators. Included in such family are heterogeneous cavitation problems of Prandtl–Batchelor type, singular degenerate elliptic equations; and obstacle type systems. The Euler–Lagrange equation associated to $\mathcal{J}_\gamma$ becomes singular along the free interface $\{u = 0\}$. The degree of singularity is, in turn, dimmed by the parameter $\gamma \in [0, 1]$. For $0 < \gamma < 1$ we show that local minima are locally of class $C^{1,\alpha}$ for a sharp α that depends on dimension, p and γ. For $\gamma = 0$ we obtain a quantitative, asymptotically optimal result, which assures that local minima are Log-Lipschitz continuous. The results proven in this article are new even in the classical context of linear, nondegenerate equations.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $2 \leqslant p < +\infty$, $f \in L^q(\Omega)$ for $q \geqslant n$ and $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, with, say, $\varphi^+ \neq 0$. The objective of the present manuscript is to derive optimal interior regularity estimates for the archetypal class of heterogeneous non-differentiable functionals

$$\mathcal{J}_\gamma(v) := \int_\Omega \left(|\nabla v|^p + F_\gamma(v) + f(X) \cdot v\right) dX \to \min, \tag{1.1}$$

among competing functions $v \in W_0^{1,p}(\Omega) + \varphi$. The parameter γ in (1.1) varies continuously from 0 to 1, i.e., $\gamma \in [0, 1]$ and the non-differentiable potential F_γ is given by

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$$F_\gamma(v) := \lambda_+\big(v^+\big)^\gamma + \lambda_-\big(v^-\big)^\gamma, \tag{1.2}$$

for scalars $0 \leqslant \lambda_- < \lambda_+ < \infty$. As usual, $v^\pm := \max\{\pm v, 0\}$, and, by convention,

$$F_0(v) := \lambda_+\chi_{\{v>0\}} + \lambda_-\chi_{\{v\leqslant 0\}}. \tag{1.3}$$

The non-differentiability of the potential F_γ impels the Euler–Lagrange equation associated to $\mathcal{J}_\gamma$ to be singular along the *a priori* unknown interface

$$\mathfrak{F}_\gamma := \big(\partial\{u_\gamma > 0\} \cup \partial\{u_\gamma < 0\}\big) \cap \Omega,$$

between the positive and negative phases of a minimum. In fact, a minimizer satisfies, in some weak sense, the following p-degenerate and singular PDE

$$\Delta_p u = \frac{\gamma}{p}\big(\lambda_+\big(u^+\big)^{\gamma-1}\chi_{\{u>0\}} - \lambda_-\big(u^-\big)^{\gamma-1}\chi_{\{u<0\}}\big) + \frac{1}{p}f(X) \quad \text{in } \Omega, \tag{1.4}$$

where $\Delta_p u$ denotes the classical p-Laplacian operator,

$$\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big).$$

The potential F_0 is actually discontinuous and that further enforces the flux balance

$$\big|\nabla u_0^+\big|^p - \big|\nabla u_0^-\big|^p = \frac{1}{p-1}(\lambda_+ - \lambda_-), \tag{1.5}$$

along the free boundary of the problem, which breaks down the continuity of the gradient through $\mathfrak{F}_0$.

A number of important mathematical problems, coming from several different contexts, are modeled by optimization setups, for which Eq. (1.1) serves as an emblematic, leading prototype. This fact has fostered massive investigations, and linear versions, $p = 2$, of the minimization problem (1.1) have indeed received overwhelming attention in the past four decades. The upper case $\gamma = 1$ is related to obstacle type problems. The linear, homogeneous, one phase obstacle problem, i.e., $p = 2$, $f(X) \equiv 0$ and $\varphi \geqslant 0$ was fully studied in the 70s by a number of leading mathematicians: Frehse, Stampacchia, Kinderlehrer, Brezis, Caffarelli, among others. It has been established that the minimum is locally of class $C^{1,1}$ and this is the optimal regularity for solution. The two-phase version of the problem, i.e., with no sign restriction on the boundary datum φ, challenged the community for over three decades. $C^{1,1}$ estimate for two-phase obstacle problems was established in [19] with the aid of the powerful *almost* monotonicity formula obtained in [5].

The lower limiting case, $\gamma = 0$, relates to jets flow and cavities problems. The linear, homogeneous, one phase version of the problem was studied in [1], where it is proven that minima are Lipschitz continuous. The two-phase version of this problem brings major new difficulties and $C^{0,1}$ local regularity of minima was proven in [2], with the aid of the revolutionary Alt–Caffarelli–Friedman monotonicity formula, developed in that very same article. Gradient estimates for two-phase cavitation type problem with bounded non-homogeneity, i.e., $p = 2$, $f \in L^\infty$, $\gamma = 0$ in (1.1), were established by Caffarelli, Jerison and Kenig with the aid of their powerful *almost* monotonicity formula, [5].

The intermediary problem $0 < \gamma < 1$ has also received great attention in the past decades. The related free boundary problem can be used, for example, to model the density of certain chemical specie, in reaction with a porous catalyst pellet. The linear, $p = 2$, one-phase, $\varphi \geqslant 0$, homogeneous, $f \equiv 0$, version of the problem (1.1) is the theme of a successful program developed in the 80s by Phillips and Alt–Phillips, [18,17,3], among others. In similar setting, Hölder continuity of the gradient of minimizers was proven by Giaquinta and Giusti [9]. Further investigations on the linear, two-phase version of this problem also require powerful monotonicity formulae in their studies, see [27].

In the mathematical analysis of variational free boundary problems as (1.1), the first major key issue to be addressed concerns the optimal regularity estimate available for a given minimum. A simple inference on the weak Euler–Lagrange equation satisfied by a minimum, Eq. (1.4) and also the flux balance (1.5) for $\gamma = 0$, revel that $\Delta_p u$ blows up along the free boundary of the problem, $\mathfrak{F}_\gamma := \partial\{u_\gamma > 0\} \cup \partial\{u_\gamma < 0\}$. Therefore, it becomes a fundamental question to understand precisely how this phenomenon affects the (lack of) smoothness properties of minima. Under such perspective, and to some extent, the theory of two-phase free boundary problems governed by non-linear, degenerate elliptic operators had hitherto been unaccessible through current literature, mainly due to the lack of monotonicity formulae in this context.