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Combination and mean width rearrangements of solutions to elliptic equations in convex sets

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Abstract

We introduce a method to compare solutions of different equations in different domains. As a consequence, we define a new kind of rearrangement which applies to solution of fully nonlinear equations $F(x, u, Du, D^2u) = 0$, not necessarily in divergence form, in convex domains and we obtain Talenti's type results for this kind of rearrangement.

Keywords: Rearrangements; Elliptic equations; Infimal convolution; Power concave envelope; Minkowski addition of convex sets

1. Introduction

Rearrangements are among the most powerful tools in analysis. Roughly speaking they manipulate the shape of an object while preserving some of its relevant geometric properties. Typically, a rearrangement of a function is performed by acting separately on each of its level sets. Probably the most famous one is the radially symmetric decreasing rearrangement, or *Schwarz symmetrization*: the *Schwarz symmetrand* of a continuous function $w \geqslant 0$ is the function $w^\star$ whose superlevel sets are concentric balls (usually centered at the origin) with the same measure as the corresponding superlevel sets of w. Notice that $w^\star$, by definition, is equidistributed with w. When applied to the study of solutions of partial differential equations with a divergence structure, this usually leads to a comparison between the solution in a generic domain and the solution of (a possibly "rearranged" version of) the same equation in a ball with the same measure of the original domain. An archetypal result of this type is the following (see [39]): let $u^\star$ be the Schwarz symmetrand of the solution u of

$$\begin{cases} \Delta u + f(x) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

and let v be the solution of

$$\begin{cases} \Delta v + f^\star(x) = 0 & \text{in } \Omega^\star, \\ v = 0 & \text{on } \partial\Omega^\star, \end{cases}$$

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where $\Omega^\star$ is the ball (centered at the origin) with the same measure as Ω, f is a non-negative function and $f^\star$ is the Schwarz symmetrand of f. Then, under suitable summability assumptions on f, it holds

$$u^\star \leqslant v \quad \text{in } \Omega^\star, \tag{2}$$

whence

$$\|u\|_{L^p(\Omega)} \leqslant \|v\|_{L^p(\Omega^\star)} \tag{3}$$

for every $p > 0$, including $p = +\infty$.

Actually Talenti's comparison principle (2)–(3) applies to more general situations and the Laplace operator in (1) can be substituted by operators like

$$\operatorname{div}\big(a_{ij}(x)u_j\big) + c(x)u$$

or even more general ones (see for instance [2–4,39–41]), but always *in divergence form*.

Here we introduce a new kind of rearrangement, which allows us to obtain comparison results similar to (2)–(3) *for very general equations, not necessarily in divergence form*, between a classical solution in a convex domain Ω and the solution in the ball $\Omega^\sharp$ with the same mean width as Ω. Recall that *the mean width* $w(\Omega)$ of Ω is defined as follows:

$$w(\Omega) = \frac{1}{n\omega_n}\int_{S^{n-1}} \big(h(\Omega,\xi) + h(\Omega,-\xi)\big)\,d\xi = \frac{2}{n\omega_n}\int_{S^{n-1}} h(\Omega,\xi)\,d\xi,$$

where $h(\Omega,\cdot)$ is the support function of Ω (then $w(\Omega,\xi) = w(\Omega,-\xi) = h(\Omega,\xi) + h(\Omega,-\xi)$ is *the width* of Ω in direction ξ or $-\xi$) and ω_n is the measure of the unit ball in $\mathbb{R}^n$. When Ω is a ball, $w(\Omega)$ simply coincides with its diameter; in the plane $w(\Omega)$ coincides with the perimeter of Ω, up to a factor π^{-1}. See Section 2 for more details, notation and definitions.

Precisely, we will deal with problems of the following type

$$\begin{cases} F\big(x,u,Du,D^2u\big) = 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\\ u > 0 & \text{in } \Omega, \end{cases} \tag{4}$$

where $F(x,t,\xi,A)$ is a continuous proper elliptic operator acting on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S_n$ and Ω is an open bounded convex subset of $\mathbb{R}^n$. Here Du and D^2u are the gradient and the Hessian matrix of the function u respectively, S_n is the set of the $n \times n$ real symmetric matrices.

We will see how, given a solution u of problem (4) and a parameter $p > 0$, it is possible to associate to u a symmetrand $u_p^\sharp$ which is defined in a ball $\Omega^\sharp$ having the same mean width as Ω. Under suitable assumptions on the operator F (see Theorem 6.6) we obtain a pointwise comparison analogous to (2) between $u_p^\sharp$ and the solution v in $\Omega^\sharp$, that is

$$u_p^\sharp \leqslant v \quad \text{in } \Omega^\sharp, \tag{5}$$

where v is the solution of

$$\begin{cases} F\big(x,v,Dv,D^2v\big) = 0 & \text{in } \Omega^\sharp,\\ v = 0 & \text{on } \partial\Omega^\sharp,\\ v > 0 & \text{in } \Omega^\sharp. \end{cases} \tag{6}$$

Then from (5) we get

$$\|u\|_{L^q(\Omega)} \leqslant \|v\|_{L^q(\Omega^\sharp)} \quad \text{for every } q \in (0,+\infty]. \tag{7}$$

The precise definition of $u_p^\sharp$ is actually quite involved and it will be given in Section 5. Here we just say that $u_p^\sharp$ is not equidistributed with u, in contrast with Schwarz symmetrization; indeed the measure of the super level sets of $u_p^\sharp$ is greater than the measure of the corresponding super level sets of u.