Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 32 (2015) 785–812

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Traveling wave solutions of Allen–Cahn equation with a fractional Laplacian

Changfeng Gui *, Mingfeng Zhao

Department of Mathematics, University of Connecticut, 196 Auditorium Road, Unit 3009, Storrs, CT 06269-3009, United States

Received 25 July 2013; received in revised form 14 March 2014; accepted 27 March 2014

Available online 9 May 2014

Abstract

In this paper, we show the existence and qualitative properties of traveling wave solutions to the Allen–Cahn equation with fractional Laplacians. A key ingredient is the estimation of the traveling speed of traveling wave solutions.

MSC: primary 35B32, 35C07, 35J20, 35R09, 35R11, 45G05, 47G10

Keywords: Traveling wave solution; Traveling speed; Allen–Cahn equation; Fractional Laplacian; Continuation method; Hamiltonian identity

1. Introduction

Front propagation is a natural phenomenon which has appeared in phase transition, chemical reaction, combustion, biological spreading, etc. The mechanism of front propagation is often the competing effects of diffusion and reaction. Traveling wave solutions are typical profiles of physical states near the propagating fronts, and therefore are of great importance in the study of reaction diffusion processes. There has been a tremendous amount of literature on traveling wave solutions in mathematics as well as in various branches of applied sciences (see [41,66,2,4,39,9,10,8,48,14] and references therein). Traveling wave solutions are essential building blocks in various phase field models, and play an important role in pattern formation and phase separation (see, e.g., [5,28,38], etc., for the classical model and [62,80,81] for nonlocal models with fractional Laplacians). Other nonlocal phase transition models and related traveling wave solutions have been studied in [33,29,6,7,88], and others, where the kernels of convolution in the nonlocal operators are bounded, and in [43,44,31] where the kernels are periodic.

In the study of front propagation, traditionally the diffusion process is quite standard and normal, in the sense that the concerned particles or objects are engaged in a Brownian motion with a uniformly changed random variable. The resulting diffusion effect on the physical state, when represented by a function mathematically, is the operation of Laplacian on this function. Therefore, the difference of various reaction diffusion systems relies on the nonlinear

* Corresponding author.
E-mail addresses: gui@math.uconn.edu (C. Gui), mingfeng.zhao@uconn.edu (M. Zhao).

http://dx.doi.org/10.1016/j.anihpc.2014.03.005
0294-1449/

reaction effect which varies in combustion, chemical reaction, phase transition, biological pattern formation, etc. In general, a typical reaction diffusion system is in the form of

$$u_t - \Delta u = f(u) \tag{1.1}$$

where $f(u)$ is a nonlinear function.

Recently, however, there has been a fast increasing number of studies on front propagation of reaction diffusion systems with an anomalous diffusion such as super diffusion, which plays important roles in various physical, chemical, biological and geological processes. (See, e.g., [75] for a brief summary and references therein.) Mathematically, such a super diffusion is related to Lévy process and may be modeled by a fractional Laplace operator $(-\Delta)^s u$ with $0 < s < 1$, whose Fourier transformation is $(2\pi|\xi|)^{2s}\widehat{u}$. (See [65] and [72], etc.) Below an exact definition of fractional Laplacians will be given.

In this paper, we study the traveling wave solutions of Allen–Cahn equation with a fractional Laplacian, where the nonlinear reaction is a bistable potential. If the front of a solution in large time propagates at constant speed, the solution is typically close to a profile depending on the distance away from the traveling fronts. Therefore we shall study only traveling wave solutions of one spatial variable, although more complicated traveling waves solutions do exist (see, e.g., [9,10,30,40,51,55–59,68,76,77,84,85,87,89,90,51] and references therein). More precisely, we are going to study the traveling wave solutions of the following reaction diffusion equation:

$$u_t(t,y) + (-\Delta)^s u(t,y) = f\big(u(t,y)\big), \quad \forall t > 0,\ y \in \mathbf{R}, \tag{1.2}$$

where $0 < s < 1$, and $f \in C^2(\mathbf{R})$ is a bistable potential satisfying

$$f(-1) = f(1) = 0, \qquad f'(-1) < 0, \qquad f'(1) < 0. \tag{1.3}$$

Let $G(u) := -\int_{-1}^{u} f(t)dt$ and t_0 be the zero in $(-1,1)$ of $f = -G'$ closest to 1, from (1.3), it is easy to see that $G(t_0) > G(1)$. We shall focus on the unbalanced case where $G(1) > G(-1) = 0$ and consider the following condition

$$G(u) > G(-1) = 0, \quad \forall u \in (-1,1) \quad \text{and} \quad f(u) < 0, \quad \forall u \in \Sigma(G) := \big\{u \in (-1,1) : G(u) \le G(1)\big\}. \tag{1.4}$$

This condition means that G at all critical points in $(-1,1)$ of f has value greater than $G(1)$.

The fractional Laplacian is often defined by Fourier transformation, for any $0 < s < 1$ and $u \in \mathcal{S}(\mathbf{R}^n)$, the Schwartz space of rapidly decaying smooth functions, the fractional Laplacian $(-\Delta)^s u$ is defined in [67] by

$$\widehat{(-\Delta)^s u}(y) = \big(2\pi|y|\big)^{2s}\hat{u}(y), \quad \forall y \in \mathbf{R}^n.$$

It is well known that equivalently we have

$$(-\Delta)^s u(y) = C_{n,s}\,\text{P.V.}\int_{\mathbf{R}^n} \frac{u(y) - u(z)}{|y - z|^{n+2s}}\,dz, \quad \forall y \in \mathbf{R}^n, \tag{1.5}$$

where $C_{n,s} = \frac{s 2^{2s}\Gamma(\frac{n+2s}{2})}{\pi^{\frac{n}{2}}\Gamma(1-s)}$ is a normalized constant. The above integral definition of fractional Laplacian can be used for more general functions, in particular, for $u \in C^2(\mathbf{R}^n)$.

Fractional Laplacian can also be defined as a Dirichlet to Neumann map. Define the n-dimensional *fractional Poisson kernel* $P^{n,s}$ as

$$P^{n,s}(x,y) = \frac{\Gamma(\frac{n+2s}{2})}{\pi^{\frac{n}{2}}\Gamma(s)} \frac{x^{2s}}{[x^2 + |y|^2]^{\frac{n+2s}{2}}}, \quad \forall (x,y) \in \mathbf{R}^+ \times \mathbf{R}^n = \mathbf{R}^{n+1}_+.$$

The *s-harmonic extension* $\bar{u}$ of $u \in C^2(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ in $\mathbf{R}^{n+1}_+$ is given by

$$\bar{u}(x,y) = P^s(x,\cdot) * u(y), \quad \forall (x,y) \in \mathbf{R}^{n+1}_+.$$

By L'Hospital's rule and the dominated convergence theorem, we can get

$$\lim_{x \searrow 0} -x^{1-2s}\bar{u}_x(x,y) = d_n(s)(-\Delta)^s u(y), \quad \forall y \in \mathbf{R}^n, \tag{1.6}$$