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A generalization of Marstrand's theorem for projections of cartesian products ☆

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Abstract

We prove the following variant of Marstrand's theorem about projections of cartesian products of sets:

Let $K_1, \dots, K_n$ be Borel subsets of $\mathbb{R}^{m_1}, \dots, \mathbb{R}^{m_n}$ respectively, and $\pi : \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n} \to \mathbb{R}^k$ be a surjective linear map. We set

$$\mathfrak{m} := \min\left\{ \sum_{i\in I} \dim_H(K_i) + \dim \pi\left(\bigoplus_{i\in I^c} \mathbb{R}^{m_i} \right),\ I \subset \{1, \dots, n\},\ I \neq \emptyset \right\}.$$

Consider the space $\Lambda_m = \{(t, O),\ t \in \mathbb{R},\ O \in SO(m)\}$ with the natural measure and set $\Lambda = \Lambda_{m_1} \times \dots \times \Lambda_{m_n}$. For every $\lambda = (t_1, O_1, \dots, t_n, O_n) \in \Lambda$ and every $x = (x^1, \dots, x^n) \in \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n}$ we define $\pi_\lambda(x) = \pi(t_1 O_1 x^1, \dots, t_n O_n x^n)$. Then we have

Theorem.

(i) *If* $\mathfrak{m} > k$*, then* $\pi_\lambda(K_1 \times \dots \times K_n)$ *has positive* k*-dimensional Lebesgue measure for almost every* $\lambda \in \Lambda$.

(ii) *If* $\mathfrak{m} \leqslant k$ *and* $\dim_H(K_1 \times \dots \times K_n) = \dim_H(K_1) + \dots + \dim_H(K_n)$*, then* $\dim_H(\pi_\lambda(K_1 \times \dots \times K_n)) = \mathfrak{m}$ *for almost every* $\lambda \in \Lambda$.

Keywords: Fractal geometry; Hausdorff dimensions; Potential theory; Fourier transform; Dynamical systems

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1. Introduction

Let us denote by $\dim_H(X)$ the Hausdorff dimension of the set X. For n and k integers with $0<k<n$, $\Pi_{n.k}$ denotes the space of orthogonal projections from $\mathbb{R}^n$ to k-dimensional subspaces of $\mathbb{R}^n$, with natural measure. A fundamental result in dimensions of projections is the following theorem:

Theorem *(Marstrand–Kaufman–Mattila). Let $E\subset\mathbb{R}^n$ a Borel set. Then:*

(i) *If $\dim_H(E)>k$, then $\pi(E)$ has positive k-dimensional Lebesgue measure for almost every $\pi\in\Pi_{n.k}$.*
(ii) *If $\dim_H(E)\leqslant k$, then $\dim_H(\pi(E))=\dim_H(E)$ for almost every $\pi\in\Pi_{n.k}$.*

This theorem was first proven for planar sets by Marstrand [3]. Marstrand's proof used geometric methods. Later, Kaufman [2] gave an alternative proof of the same result applying potential-theoretic methods. Finally, Mattila [4] generalized it to higher dimensions; his proof combines Marstrand and Kaufman methods.

There are other variants of Marstrand–Mattila's theorem that were unified in a more general result due to Peres and Schlag [7]. These authors studied general smooth families of projections, using some methods from harmonic analysis. The crucial characteristic that is common to all families of projections considered in Peres–Schlag's result is a transversality property (see [7, Definition 7.2]).

We are interested in Marstrand's projection result that actually is outside of Peres–Schlag's scheme (the families of projections considered here, in general, are not transversal). This result was motivated by the problem of understanding the behavior of projections of cartesian products of sets, by a fixed projection map.

Let $K_1,\dots,K_n$ be Borel subsets of $\mathbb{R}^{m_1},\dots,\mathbb{R}^{m_n}$ respectively, and $\pi:\mathbb{R}^{m_1}\times\dots\times\mathbb{R}^{m_n}\to\mathbb{R}^k$ be a linear map. Then

$$\dim_H\big(\pi(K_1\times\dots\times K_n)\big)\leqslant\min\left\{\sum_{i\in I}\dim_H(K_i)+\dim\pi\left(\bigoplus_{i\in I^c}\mathbb{R}^{m_i}\right),\ I\subset\{1,\dots,n\}\right\},\tag{1.1}$$

with the conventions $\sum_{i\in\emptyset}\dim_H(K_i)=0$, $\dim\emptyset=0$.

Consider the space $\Lambda_m=\{(t,O),\ t\in\mathbb{R},\ O\in SO(m)\}$ with the natural measure and set $\Lambda=\Lambda_{m_1}\times\dots\times\Lambda_{m_n}$. For every $x=(x^1,\dots,x^n)\in\mathbb{R}^{m_1}\times\dots\times\mathbb{R}^{m_n}$ and every $\lambda=(t_1,O_1,\dots,t_n,O_n)\in\Lambda$ we define $\pi_\lambda(x)=\pi(t_1O_1x^1,\dots,t_nO_nx^n)$. Suppose that π is surjective and set

$$\mathfrak{m}:=\min\left\{\sum_{i\in I}\dim_H(K_i)+\dim\pi\left(\bigoplus_{i\in I^c}\mathbb{R}^{m_i}\right),\ I\subset\{1,\dots,n\},\ I\neq\emptyset\right\}.$$

Then we have

Theorem 1.1.

(i) *If $\mathfrak{m}>k$, then $\pi_\lambda(K_1\times\dots\times K_n)$ has positive k-dimensional Lebesgue measure for almost every $\lambda\in\Lambda$.*
(ii) *If $\mathfrak{m}\leqslant k$ and $\dim_H(K_1\times\dots\times K_n)=\dim_H(K_1)+\dots+\dim_H(K_n)$, then $\dim_H(\pi_\lambda(K_1\times\dots\times K_n))=\mathfrak{m}$ for almost every $\lambda\in\Lambda$.*

We recover Marstrand–Mattila's theorem considering the cartesian product of only one set.

Theorem 2.3 is a fundamental tool in our forthcoming work which generalizes the result of Moreira and Yoccoz [6] about stable intersections of two regular Cantor sets for projections of cartesian products of several regular Cantor sets. We prove the following result: for any given surjective linear map $\pi:\mathbb{R}^n\to\mathbb{R}^k$, typically for regular Cantor sets on the real line $K_1,\dots,K_n$ with $\mathfrak{m}>k$, the set $\pi(K_1\times\dots\times K_n)$ persistently contains non-empty open sets of $\mathbb{R}^k$. Such a result in particular implies an analogous result for simultaneous stable intersections of several regular Cantor sets on the real line.

In another forthcoming work, in collaboration with Pablo Shmerkin, we use the results of this paper combined with the techniques in [1] in order to obtain exact formulas for the Hausdorff dimensions of projections of cartesian products of (real or complex) regular Cantor sets under explicit irrationality conditions.