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A critical fractional equation with concave–convex power nonlinearities ☆

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Abstract

In this work we study the following fractional critical problem

$$(P_\lambda) = \begin{cases} (-\Delta)^s u = \lambda u^q + u^{2_s^*-1}, \quad u>0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a regular bounded domain, $\lambda > 0$, $0 < s < 1$ and $n > 2s$. Here $(-\Delta)^s$ denotes the fractional Laplace operator defined, up to a normalization factor, by

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy, \quad x \in \mathbb{R}^n.$$

Our main results show the existence and multiplicity of solutions to problem (P_λ) for different values of λ. The dependency on this parameter changes according to whether we consider the concave power case ($0 < q < 1$) or the convex power case ($1 < q < 2_s^* - 1$). These two cases will be treated separately.

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1. Introduction

In recent years, considerable attention has been given to nonlocal diffusion problems, in particular to the ones driven by the fractional Laplace operator. One of the reasons for this comes from the fact that this operator naturally arises in several physical phenomena like flames propagation and chemical reactions of liquids, in population dynamics and geophysical fluid dynamics, or in mathematical finance (American options). It also provides a simple model to describe certain jump Lévy processes in probability theory. In all these cases, the nonlocal effect is modeled by the singularity at infinity. For more details and applications, see [6,9,20,27,49,50] and the references therein.

In this paper we focus our attention on critical nonlocal fractional problems. To be more precise, we consider the following critical problem with convex–concave nonlinearities

$$(P_\lambda) = \begin{cases} (-\Delta)^s u = \lambda u^q + u^{2_s^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a regular bounded domain, $\lambda > 0$, $n > 2s$, $0 < q < 2_s^* - 1$ and

$$2_s^* = \frac{2n}{n-2s} \tag{1.1}$$

is the fractional critical Sobolev exponent. Here $(-\Delta)^s$ is the fractional Laplace operator defined, up to a normalization factor, by the Riesz potential as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy, \qquad x \in \mathbb{R}^n, \tag{1.2}$$

where $s \in (0, 1)$ is a fixed parameter (see [46, Chapter 5] or [22,45] for further details).

One can also define a fractional power of the Laplacian using spectral decomposition. The same problem considered here but for this spectral fractional Laplacian has been treated in [7]. Some related problems involving this operator have been studied in [11,14,19,48]. As in [7] the purpose of this paper is to study the existence of weak solutions for (P_λ). Previous works related to the operator defined in (1.2), or by a more general kernel, can be found in [16,23,30,34,35,37,38,41–43].

Problems similar to (P_λ) have been also studied in the local setting with different elliptic operators. As far as we know, the first example in this direction was given in [25] for the p-Laplacian operator. Other results, this time for the Laplacian (or essentially the classical Laplacian) operator can be found in [1,4,10,18]. More generally, the case of fully nonlinear operators has been studied in [17].

It is worth noting here that the problem (P_λ), with $\lambda = 0$, has no solution whenever Ω is a star-shaped domain. This has been proved in [24,36] using a Pohozaev identity for the operator $(-\Delta)^s$. This fact motivates the perturbation term λu^q, $\lambda > 0$, in our work.

We now summarize the main results of the paper. First, in Section 2 we look at the problem (P_λ) in the concave case $q < 1$ and prove the following.

Theorem 1.1. *Assume $0 < q < 1$, $0 < s < 1$, and $n > 2s$. Then, there exists $0 < \Lambda < \infty$ such that problem (P_λ)*

(1) *has no solution for $\lambda > \Lambda$;*
(2) *has a minimal solution for any $0 < \lambda < \Lambda$; moreover, the family of minimal solutions is increasing with respect to λ;*
(3) *if $\lambda = \Lambda$ there exists at least one solution;*
(4) *for $0 < \lambda < \Lambda$ there are at least two solutions.*

The convex case is treated in Section 3. The existence result for problem (P_λ) is given by:

Theorem 1.2. *Assume $1 < q < 2_s^* - 1$, $0 < s < 1$, and $n > 2s$. Then, problem (P_λ) admits at least one solution provided that either*