Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 33 (2016) 965–1007
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

A semilinear singular Sturm–Liouville equation involving measure data ☆

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Received 21 September 2012; received in revised form 24 March 2013; accepted 28 March 2013

Available online 20 March 2015

Abstract

Given $\alpha > 0$ and $p > 1$, let μ be a bounded Radon measure on the interval $(-1, 1)$. We are interested in the equation $-(|x|^{2\alpha}u')' + |u|^{p-1}u = \mu$ on $(-1, 1)$ with boundary condition $u(-1) = u(1) = 0$. We establish some existence and uniqueness results. We examine the limiting behavior of three approximation schemes. The isolated singularity at 0 is also investigated.

MSC: 34B16; 34E99

Keywords: Singular Sturm–Liouville equation; Semilinear equation; Radon measure; Elliptic regularization; Classification of singularity

1. Introduction

In this paper, we consider the following semilinear singular Sturm–Liouville equation

$$\begin{cases} -(|x|^{2\alpha}u')' + |u|^{p-1}u = \mu & \text{on } (-1,1), \\ u(-1) = u(1) = 0. \end{cases} \tag{1.1}$$

Here we assume that $\alpha > 0$, $p > 1$, and $\mu \in \mathcal{M}(-1, 1)$, where $\mathcal{M}(-1, 1)$ is the space of bounded Radon measures on the interval $(-1, 1)$. We denote

$$C_0[-1,1] = \{\zeta \in C[-1,1];\ \zeta(-1) = \zeta(1) = 0\}.$$

Then μ can be viewed as a bounded linear functional on $C_0[-1, 1]$. That is,

$$\mathcal{M}(-1,1) = (C_0[-1,1])^*.$$

In the previous work [23], we studied the corresponding linear equation (i.e., $p = 1$ in (1.1)). For the linear case, we defined a notion of *solution* for all $\alpha > 0$ and a notion of *good solution* for $0 < \alpha < 1$. We proved the existence and

☆ Partially supported by the NSF Grants DMS 1207793 and the ITN "FIRST" of the Seventh Framework Programme of the European Community (grant agreement number 238702).

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http://dx.doi.org/10.1016/j.anihpc.2015.03.001

uniqueness of the good solution for every measure μ when $0 < \alpha < 1$ and we proved the uniqueness of the solution when $\alpha \geq 1$. We also presented a necessary and sufficient condition on μ for the existence of the solution when $\alpha \geq 1$.

For the semilinear equation (1.1), we can adapt from [23] the notion of solution and the notion of good solution. Rewrite (1.1) as $-(|x|^{2\alpha}u')' + u = u - |u|^{p-1}u + \mu$. Then according to [23], a function u is a *solution* of (1.1) if

$$u \in L^p(-1,1) \cap W^{1,1}_{loc}([-1,1]\setminus\{0\}),\ |x|^{2\alpha}u' \in BV(-1,1), \tag{1.2}$$

and u satisfies (1.1) in the usual sense (i.e., in the sense of measures). When $0 < \alpha < 1$, a solution u of (1.1) is called a *good solution* if it satisfies in addition

$$\begin{cases} \lim\limits_{x\to 0^+} u(x) = \lim\limits_{x\to 0^-} u(x), \text{ when } 0 < \alpha < \frac{1}{2}, \\ \lim\limits_{x\to 0^+} \left(1 + \ln\frac{1}{|x|}\right)^{-1} u(x) = \lim\limits_{x\to 0^-} \left(1 + \ln\frac{1}{|x|}\right)^{-1} u(x), \text{ when } \alpha = \frac{1}{2}, \\ \lim\limits_{x\to 0^+} |x|^{2\alpha-1}u(x) = \lim\limits_{x\to 0^-} |x|^{2\alpha-1}u(x), \text{ when } \frac{1}{2} < \alpha < 1. \end{cases} \tag{1.3}$$

In this work, we are interested in the question of existence and uniqueness, the limiting behavior of three different approximation schemes, and the classification of the isolated singularity at 0.

It turns out that we need to investigate the following four cases separately:

$$0 < \alpha \leq \frac{1}{2},\ p > 1, \tag{1.4}$$

$$\frac{1}{2} < \alpha < 1,\ 1 < p < \frac{1}{2\alpha - 1}, \tag{1.5}$$

$$\frac{1}{2} < \alpha < 1,\ p \geq \frac{1}{2\alpha - 1}, \tag{1.6}$$

$$\alpha \geq 1,\ p > 1. \tag{1.7}$$

As we are going to see, the notion of good solution is only necessary for cases (1.4) and (1.5). In fact, for the case (1.6), if the solution exists, it must be the good solution.

Our first result concerns the question of uniqueness.

Theorem 1.1. *If α and p satisfy (1.4) or (1.5), then for every $\mu \in \mathcal{M}(-1,1)$ there exists at most one good solution of (1.1). If α and p satisfy (1.6) or (1.7), then for every $\mu \in \mathcal{M}(-1,1)$ there exists at most one solution of (1.1).*

Remark 1.1. In fact, for α and p satisfying (1.4) or (1.5), there exist infinitely many solutions of (1.1); all of them will be identified in Section 7.

The next two theorems answer the question of existence.

Theorem 1.2. *Assume that α and p satisfy (1.4) or (1.5). For every $\mu \in \mathcal{M}(-1,1)$, there exists a (unique) good solution of (1.1). Moreover, the good solution satisfies*

(i) $\lim\limits_{x\to 0} \left(1 + \ln\frac{1}{|x|}\right)^{-1} u(x) = -\lim\limits_{x\to 0^+} |x|u'(x) = \lim\limits_{x\to 0^-} |x|u'(x) = \frac{\mu(\{0\})}{2}$ *when $\alpha = \frac{1}{2}$ and $p > 1$,*

(ii) $\lim\limits_{x\to 0} |x|^{2\alpha-1}u(x) = -\lim\limits_{x\to 0^+} \frac{|x|^{2\alpha}u'(x)}{2\alpha-1} = \lim\limits_{x\to 0^-} \frac{|x|^{2\alpha}u'(x)}{2\alpha-1} = \frac{\mu(\{0\})}{4\alpha-2}$ *when $\frac{1}{2} < \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$,*

(iii) $\left\| |u|^{p-1}u - |\hat{u}|^{p-1}\hat{u} \right\|_{L^1} \leq \|\mu - \hat{\mu}\|_{\mathcal{M}}$ *and* $\left\| \left(|u|^{p-1}u - |\hat{u}|^{p-1}\hat{u}\right)^+ \right\|_{L^1} \leq \left\| (\mu - \hat{\mu})^+ \right\|_{\mathcal{M}}$, *for $\mu, \hat{\mu} \in \mathcal{M}(-1,1)$ and their corresponding good solutions $u, \hat{u}$.*

Theorem 1.3. *Assume that α and p satisfy (1.6) or (1.7). For each $\mu \in \mathcal{M}(-1,1)$, there exists a (unique) solution of (1.1) if and only if $\mu(\{0\}) = 0$. Moreover, if the solution exists, it satisfies*