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Global bifurcation theory for periodic traveling interfacial gravity–capillary waves

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Abstract

We consider the global bifurcation problem for spatially periodic traveling waves for two-dimensional gravity–capillary vortex sheets. The two fluids have arbitrary constant, non-negative densities (not both zero), the gravity parameter can be positive, negative, or zero, and the surface tension parameter is positive. Thus, included in the parameter set are the cases of pure capillary water waves and gravity–capillary water waves. Our choice of coordinates allows for the possibility that the fluid interface is not a graph over the horizontal. We use a technical reformulation which converts the traveling wave equations into a system of the form "identity plus compact." Rabinowitz' global bifurcation theorem is applied and the final conclusion is the existence of either a closed loop of solutions, or an unbounded set of nontrivial traveling wave solutions which contains waves which may move arbitrarily fast, become arbitrarily long, form singularities in the vorticity or curvature, or whose interfaces self-intersect.

Keywords: Global bifurcation; Surface tension; Traveling wave; Interfacial flow; Water wave

1. Introduction

We consider the case of two two-dimensional fluids, of infinite vertical extent and periodic in the horizontal direction (of period $M > 0$) and separated by an interface which is free to move. Each fluid has a constant, non-negative density: $\rho_2 \geq 0$ in the upper fluid and $\rho_1 \geq 0$ in the lower. Of course, we do not allow both densities to be zero, but if one of the densities is zero, then it is known as the water wave case. The velocity of each fluid satisfies the incompressible, irrotational Euler equations. The restoring forces in the problem include non-zero surface tension (with surface tension constant $\tau > 0$) on the interface and a gravitational body force (with acceleration $g \in \mathbf{R}$, possibly

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zero) which acts in the vertical direction. Since the fluids are irrotational, the interface is a vortex sheet, meaning that the vorticity in the problem is an amplitude times a Dirac mass supported on the interface. We call this problem "the two-dimensional gravity–capillary vortex sheet problem." The average vortex strength on the interface is denoted by $\bar{\gamma}$.

In [2], two of the authors and Akers established a new formulation for the traveling wave problem for parameterized curves, and applied it to the vortex sheet with surface tension (in case the two fluids have the same density). The curves in [2] may have multi-valued height. This is significant since it is known that there exist traveling waves in the presence of surface tension which do indeed have multi-valued height; the most famous such waves are the Crapper waves [14], and there are other, related waves known [24,4,15]. The results of [2] were both analytical and computational; the analytical conclusion was a local bifurcation theorem, demonstrating that there exist traveling vortex sheets with surface tension nearby to equilibrium. In the present work, we establish a *global* bifurcation theorem for the problem with general densities. We now state a somewhat informal version of this theorem:

Theorem 1 *(Main theorem). For all choices of the constants $\tau > 0$, $M > 0$, $\bar{\gamma} \in \mathbf{R}$, $\rho_1, \rho_2 \geq 0$ (not both zero) and $g \in \mathbf{R}$, there exist a countable number of connected sets of smooth[4] non-trivial symmetric periodic traveling wave solutions, bifurcating from a quiescent equilibrium, for the two-dimensional gravity–capillary vortex sheet problem. If $\bar{\gamma} \neq 0$ or $\rho_1 \neq \rho_2$, then each of these connected sets has at least one of the following properties:*

(a) *it contains waves whose interfaces have lengths per period which are arbitrarily long;*
(b) *it contains waves whose interfaces have arbitrarily large curvature;*
(c) *it contains waves where the jump of the tangential component of the fluid velocity across the interface or its derivative is arbitrarily large;*
(d) *its closure contains a wave whose interface has a point of self-intersection;*
(e) *it contains a sequence of waves whose interfaces converge to a flat configuration but whose speeds contain at least two convergent subsequences whose limits differ.*

In the case that $\bar{\gamma} = 0$ and $\rho_1 = \rho_2$, each connected set has at least one of the properties (a)–(f), where (f) is the following:

(f) *it contains waves which have speeds which are arbitrarily large.*

We mention that in the case of pure gravity waves, it has sometimes been possible to rule out the possibility of an outcome like (e) above; one such paper, for example, is [11]. The argument to eliminate such an outcome is typically a maximum principle argument, and this type of argument appears to be unavailable in the present setting because of the larger number of derivatives stemming from the presence of surface tension. In a forthcoming numerical work, computations will be presented which indicate that in some cases, outcome (e) can in fact occur for gravity–capillary waves [3].

Following [2], we start from the formulation of the problem introduced by Hou, Lowengrub, and Shelley, which uses geometric dependent variables and a normalized arclength parameterization of the free surface [19,20]. This formulation follows from the observation that the tangential velocity can be chosen arbitrarily, while only the normal velocity needs to be chosen in accordance with the physics of the problem. The tangential velocity can then be selected in a convenient fashion which allows us to specialize the equations of motion to the periodic traveling wave case in a way that does not require the interface to be a graph over the horizontal coordinate. The resulting equations are non-local, nonlinear and involve the singular Birkhoff–Rott integral. Despite their complicated appearance, using several well-known properties of the Birkhoff–Rott integral we are able to recast the traveling wave equations in the form of "identity plus compact." Consequently, we are able to use an abstract version of the Rabinowitz global-bifurcation theory [29] to prove our main result. An interesting feature of our formulation is that, unlike similar formulations that allow for overturning waves by using a conformal mapping, an extension of the present method to the case of 3D waves, using for instance ideas like those in [6], seems entirely possible.

[4] Here and below, when we say a function is "smooth" we mean that its derivatives of all orders exist.