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Parabolic limit with differential constraints of first-order quasilinear hyperbolic systems

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Abstract

The goal of this work is to provide a general framework to study singular limits of initial-value problems for first-order quasilinear hyperbolic systems with stiff source terms in several space variables. We propose structural stability conditions of the problem and construct an approximate solution by a formal asymptotic expansion with initial layer corrections. In general, the equations defining the approximate solution may come together with differential constraints, and so far there are no results for the existence of solutions. Therefore, sufficient conditions are shown so that these equations are parabolic without differential constraint. We justify rigorously the validity of the asymptotic expansion on a time interval independent of the parameter, in the case of the existence of approximate solutions. Applications of the result include Euler equations with damping and an Euler–Maxwell system with relaxation. The latter system was considered in [27,9] which contain ideas used in the present paper.

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1. Introduction

This work is concerned with singular limits of first-order quasilinear hyperbolic equations with stiff source terms of the form

$$\partial_t U + \frac{1}{\varepsilon}\sum_{j=1}^{d} A_j(U)\partial_{x_j} U = \frac{Q(\varepsilon, U)}{\varepsilon^2}, \tag{1.1}$$

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with initial conditions

$$U(0,x) = \bar{U}(x,\varepsilon). \tag{1.2}$$

Here $U : \mathbb{R}_t^+ \times \mathbb{R}_x^d \longrightarrow G \subset \mathbb{R}^n$ is the unknown variable with $x = (x_1, \cdots, x_d)$, $\varepsilon \in (0,1]$ is a small parameter and $Q : [0,1] \times G \longrightarrow \mathbb{R}^n$ is a smooth vector function. In physical models, ε often stands for a relaxation time. The set G is called the state space and A_j $(1 \le j \le d)$ are $n \times n$ smooth matrix functions defined on G. We suppose that (1.1) is symmetrizable hyperbolic (see [8]): i.e., there exists a symmetric positive definite matrix $A_0(U)$, called symmetrizer, such that for all $U \in G$,

(i) $A_0(U)\xi \cdot \xi \ge M_0|\xi|^2$, for all $\xi \in \mathbb{R}^n$;
(ii) $\tilde{A}_j(U) \stackrel{def}{=} A_0(U)A_j(U)$ is symmetric for all $1 \le j \le d$,

where $M_0 > 0$ is a constant, "$\cdot$" is the inner product of $\mathbb{R}^n$ and $|\cdot|$ is the Euclidean norm of $\mathbb{R}^n$. In general, Q only depends on U. The fact that it may also depend on ε is due to an Euler–Maxwell system with relaxation (see the last section).

In (1.1), the variable t should be understood as a slow time linked with the usual time t' by $t = \varepsilon t'$. Therefore, (1.1) is equivalent to

$$\partial_{t'} U + \sum_{j=1}^{d} A_j(U)\partial_{x_j} U = \frac{Q(\varepsilon, U)}{\varepsilon}. \tag{1.3}$$

System (1.3) is a general form of first-order quasilinear hyperbolic equations with stiff relaxation source terms. It was studied by many authors in the case where Q is a function of only U. Under stability conditions, the limit equations of (1.3) as $\varepsilon \to 0$ are of first-order hyperbolic type. For mathematical results and physical examples of (1.3), we refer to [32,19,6,12,3,25,26,33,29,34] and references therein.

The aim of the present work is to study the limit of smooth solutions of (1.1)–(1.2) as $\varepsilon \to 0$, in a d-dimensional torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$. Then $\bar{U}$ is supposed to be smooth and periodic with respect to x. As usual for first-order hyperbolic problems with relaxation, we assume

$$Q(0,U) = \begin{bmatrix} 0 \\ q(U) \end{bmatrix}, \tag{1.4}$$

where $q : G \longrightarrow \mathbb{R}^r$ is a smooth function, $1 \le r \le n$. With the same partition, we denote

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad u \in \mathbb{R}^{n-r}, \ v \in \mathbb{R}^r.$$

More generally, a vector $V \in \mathbb{R}^n$ and an $n \times n$ matrix M will be denoted by $\begin{bmatrix} V^I \\ V^{II} \end{bmatrix}$ and $\begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix}$, respectively. In order to obtain a parabolic limit from (1.1), we further assume

$$q(U) = 0 \Longleftrightarrow v = 0, \text{ and } \partial_v q(u,0) \text{ is invertible for all } u \in \mathbb{R}^{n-r}. \tag{1.5}$$

The singular limit problem $\varepsilon \to 0$ for (1.1) was considered in [22,21,23] in the case of special models. See also [4] for the approximation of parabolic equations by diffusive BGK models. Contrarily to (1.3), in general the limit equations of (1.1) are of parabolic type. In [16], Lattanzio and Yong considered a first-order symmetrizable hyperbolic system of the form

$$\partial_t U + \frac{1}{\varepsilon}\sum_{j=1}^{d} A_j(\varepsilon U)\partial_{x_j} U + \sum_{j=1}^{d} \bar{A}_j(U)\partial_{x_j} U = \frac{Q(U)}{\varepsilon^2}, \tag{1.6}$$

with smooth and periodic initial data $\bar{U}$ given in (1.2). Assuming appropriate stability conditions and the existence of approximate solutions, they proved the convergence of the system to parabolic type equations on a time interval independent of ε. In this problem, the singular limit arises from $Q(U)/\varepsilon^2$ and the term containing $A_j(\varepsilon U)/\varepsilon$, and