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# Self-similar solutions with fat tails for Smoluchowski's coagulation equation with singular kernels

# Solutions auto-similaire avec queues lourdes pour l'équation de coagulation de Smoluchowski avec noyaux singuliers

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**Abstract**

We show the existence of self-similar solutions with fat tails for Smoluchowski's coagulation equation for homogeneous kernels satisfying $C_1\left(x^{-a}y^b + x^b y^{-a}\right) \leq K\left(x, y\right) \leq C_2\left(x^{-a}y^b + x^b y^{-a}\right)$ with $a > 0$ and $b < 1$. This covers especially the case of Smoluchowski's classical kernel $K(x, y) = (x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3})$.

For the proof of existence we take a self-similar solution $h_\varepsilon$ for a regularized kernel $K_\varepsilon$ and pass to the limit $\varepsilon \to 0$ to obtain a solution for the original kernel $K$. The main difficulty is to establish a uniform lower bound on $h_\varepsilon$. The basic idea for this is to consider the time-dependent problem and to choose a special test function that solves the dual problem.



**Résumé**

Nous démontrons l'existence des solutions auto-similaires avec queues lourdes pour l'équation de coagulation de Smoluchowski avec un noyau $K$ satisfaisant $C_1\left(x^{-a}y^b + x^b y^{-a}\right) \leq K\left(x, y\right) \leq C_2\left(x^{-a}y^b + x^b y^{-a}\right)$ avec $a > 0$ et $b < 1$. Cela contient en particulier le noyau classique de Smoluchowski $K(x, y) = (x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3})$.

Pour la démonstration de l'existence nous prenons une solution auto-similaire $h_\varepsilon$ pour un noyau régularisé $K_\varepsilon$ et nous obtenons une solution pour le noyau original $K$ en passant à la limite $\varepsilon \to 0$. La difficulté principale consiste à établir une borne inférieure

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pour $h_\varepsilon$. La clé ici est de considérer le problème dépendant du temps et choisir une solution du problème dual comme fonction test dans la formulation faible de l'équation auto-similaire.


*Keywords:* Smoluchowski's coagulation equation; Self-similar solution; Singular kernel; Fat tail; Dual problem

## 1. Introduction

### *1.1. Smoluchowski's equation and self-similarity*

Smoluchowski's coagulation equation [13] describes irreversible aggregation of clusters through binary collisions by a mean-field model for the density $f(\xi,t)$ of clusters of mass $\xi$. It is assumed that the rate of coagulation of clusters of size $\xi$ and $\eta$ is given by a rate kernel $K = K(\xi,\eta)$, such that the evolution of $f$ is determined by

$$\partial_t f(\xi,t) = \frac{1}{2}\int_0^{\xi} K(\xi-\eta,\eta) f(\xi-\eta,t) f(\eta,t)\,\mathrm{d}\eta - f(\xi,t)\int_0^{\infty} K(\xi,\eta) f(\eta,t)\,\mathrm{d}\eta. \tag{1}$$

Applications in which this model has been used are numerous and include, for example, aerosol physics, polymerization, astrophysics and mathematical biology (see e.g. [1,3]).

A topic of particular interest in the theory of coagulation is the scaling hypothesis on the long-time behaviour of solutions to (1). Indeed, for homogeneous kernels one expects that solutions converge to a uniquely determined self-similar profile. This issue is however only well-understood for the solvable kernels $K(x,y)=2$, $K(x,y)=x+y$ and $K(x,y)=xy$. In these cases it is known [9] that (1) has one fast-decaying self-similar solution with finite mass and a family of so-called fat-tail or heavy-tailed self-similar solutions with power-law decay. Such solutions with certain infinite moments have been studied extensively in probability theory and are of considerable interest since they predict a high probability of the occurrence of extreme events. Furthermore, in [9] the domains of attraction of all these self-similar solutions under the evolution (1) have been completely characterized. For non-solvable kernels much less is known and it is exclusively for the case of kernels with homogeneity $\gamma < 1$. In [5,6] existence of self-similar solutions with finite mass has been established for a large range of kernels and some properties of those solutions have been investigated in [2,4,7]. More recently, the first existence results of self-similar solutions with fat tails have been proved, first for the diagonal kernel [11], then for kernels that are bounded by $C(x^\gamma + y^\gamma)$ for $\gamma \in [0,1)$ [12]. It is the goal of this paper to extend the results in [12] to singular kernels, such as Smoluchowski's classical kernel $K(x,y) = (x^{1/3}+y^{1/3})(x^{-1/3}+y^{-1/3})$.

In order to describe our results in more detail, we first derive the equation for self-similar solutions. Such solutions to (1) for kernels of homogeneity $\gamma < 1$ are of the form

$$f(\xi,t) = \frac{\beta}{t^\alpha} g(x), \qquad \alpha = 1 + (1+\gamma)\beta, \qquad x = \frac{\xi}{t^\beta}, \tag{2}$$

where the self-similar profile $g$ solves

$$-\frac{\alpha}{\beta} g - x g'(x) = \frac{1}{2}\int_0^{x} K(x-y,y) g(x-y) g(y)\,\mathrm{d}y - g(x)\int_0^{\infty} K(x,y) g(y)\,\mathrm{d}y. \tag{3}$$

It is known that for some kernels the self-similar profiles are singular at the origin, so that the integrals on the right-hand side are not finite and it is necessary to rewrite the equation in a weaker form. Multiplying the equation by $x$ and rearranging we obtain that a weak self-similar solution $g$ solves

$$\partial_x(x^2 g(x)) = \partial_x\left[\int_0^{x}\int_{x-y}^{\infty} y K(y,z) g(z) g(y)\,\mathrm{d}z\,\mathrm{d}y\right] + \left((1-\gamma) - \frac{1}{\beta}\right) x g(x) \tag{4}$$